

AIRY PROCESSES AND VARIATIONAL PROBLEMS

JEREMY QUASTEL AND DANIEL REMENIK

ABSTRACT. We review the Airy processes; their formulation and how they are conjectured to govern the large time, large distance spatial fluctuations of one dimensional random growth models. We also describe formulas which express the probabilities that they lie below a given curve as Fredholm determinants of certain boundary value operators, and the several applications of these formulas to variational problems involving Airy processes that arise in physical problems, as well as to their local behaviour.

CONTENTS

1. Introduction	1
1.1. Airy processes and the KPZ universality class	1
1.2. Directed random polymers and last passage percolation	2
1.3. The continuum random polymer and the stochastic heat equation	11
1.4. General conjectural picture for the SHE	13
1.5. Determinantal formulas and extended kernels	15
2. Fredholm Determinants	19
3. Boundary Value Kernels and Continuum Statistics of Airy Processes	23
3.1. Boundary value kernel formulas for finite-dimensional distributions	23
3.2. Continuum statistics and boundary value problems	25
4. Applications	26
4.1. Point-to-line LPP and GOE	27
4.2. Endpoint distribution of directed polymers	29
4.3. Local behavior of Airy_1	35
4.4. Marginals of $\text{Airy}_{2 \rightarrow 1}$	36
References	38

1. INTRODUCTION

1.1. Airy processes and the KPZ universality class. The *Airy processes* are a collection of stochastic processes which are expected to govern the long time, large scale, spatial fluctuations of random growth models in the one dimensional *Kardar-Parisi-Zhang (KPZ) universality class* for wide classes of initial data. Although there is no precise definition of the KPZ class, it can be identified at the roughest level by the unusual $t^{1/3}$ scale of fluctuations. It is expected to contain a large class of random growth processes, as well as randomly stirred one dimensional fluids, polymer chains directed in one dimension and fluctuating transversally in the other due to a random potential (with applications to domain interfaces in disordered crystals), driven lattice gas models, reaction-diffusion models in two-dimensional random media (including biological models such as bacterial colonies), randomly forced Hamilton-Jacobi equations, etc. The model giving its name to the universality class is the *KPZ equation*, which was introduced by Kardar, Parisi, and Zhang

[KPZ86] as a model of randomly growing interfaces, and is given by

$$\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \xi,$$

where $\xi(t, x)$ is Gaussian space-time white noise, $\mathbb{E}(\xi(t, x)\xi(s, y)) = \delta_{s=t}\delta_{x=y}$.

A combination of non-rigorous methods (renormalization, mode-coupling, replicas) and mathematical breakthroughs on a few special models has led to very precise predictions of universal scaling exponents and exact statistical distributions describing the long time properties. These predictions have been repeatedly confirmed through Monte-Carlo simulation as well as experiments; in particular, recent spectacular experiments on turbulent liquid crystals by Takeuchi and Sano [TS10; TS12] have been able to even confirm some of the predicted fluctuation statistics in a physical system.

The conjectural picture that has developed is that the universality class is divided into subuniversality classes which depend on the class of initial data (or boundary conditions), but not on other details of the particular models. There are three classes of initial data which stand out because of their self-similarity properties: Dirac δ_0 , corresponding to curved, or droplet type initial data; 0, corresponding to growth off a flat substrate; and $e^{B(x)}$ where $B(x)$ is a two sided Brownian motion, corresponding to growth in equilibrium. As we will see later, each of these three classes correspond to concrete initial (or boundary) conditions for the discrete models in the KPZ class. In addition to these three basic initial data, there are three non-homogeneous subuniversality classes corresponding roughly to starting with one of the basic three on one side of the origin, and another on the other side. For one specific discrete model (last passage percolation or, equivalently, the totally asymmetric exclusion process) the asymptotic spatial fluctuations have been computed exactly for these six basic classes of initial data, and are given by the Airy processes: the three basic Airy processes, Airy_2 , Airy_1 and $\text{Airy}_{\text{stat}}$, and the crossover Airy processes $\text{Airy}_{2 \rightarrow 1}$, $\text{Airy}_{2 \rightarrow \text{BM}}$ and $\text{Airy}_{1 \rightarrow \text{BM}}$. Although these processes have been proved to arise as the limiting spatial fluctuations only for one model (and actually several others in the case of the Airy_2 process), as a consequence of the universality conjecture for the KPZ class it is expected that the same should hold for the other models in the class.

The purpose of this review is two-fold. First, we will explain in detail in the introduction the conjectural picture that we have just sketched from two different points of view: last passage percolation (or, more generally, directed random polymers) and the KPZ equation (or, more precisely, the stochastic heat equation). Along the way we will survey known results for these models.

Our second purpose is to survey a collection of results for the Airy processes which express the probability that they lie below a given curve as Fredholm determinants of certain boundary value operators. These expressions have turned out to be very useful in obtaining some exact distributions through certain variational formulas, and in addition have allowed one to study some local properties of these processes. This will be the subject of Sections 2-4,

Acknowledgments. JQ was supported by the Natural Science and Engineering Research Council of Canada. DR was supported by Fondecyt Grant 1120309 and Conicyt Basal-CMM. The authors thank an anonymous referee for many useful suggestions.

1.2. Directed random polymers and last passage percolation.

1.2.1. Polymers. Consider the following model of a *directed polymer in a random environment*. A *polymer path* is an up-right path $\pi = (\pi_0, \pi_1, \dots)$ in $(\mathbb{Z}_+)^2$ started at the origin, that is, $\pi_0 = (0, 0)$ and $\pi_k - \pi_{k-1} \in \{(1, 0), (0, 1)\}$ (see Figure 1a). On $(\mathbb{Z}_+)^2$ we place a collection of independent random weights $\{\omega_{i,j}\}_{i,j \geq 0}$. The *energy* of a polymer path

segment π of length N is

$$H_N(\pi) = - \sum_{k=1}^N \omega_{\pi_k}.$$

We define the *weight* of such a polymer path segment as

$$(1.1) \quad W_N(\pi) = e^{-\beta H_N(\pi)} = e^{\beta \sum_{k=1}^N \omega_{\pi_k}}$$

for some fixed $\beta > 0$ which is known as the *inverse temperature* of the model. Let $\Pi_{M,N}$ denote the set of up-right paths going from the origin to $(M, N) \in (\mathbb{Z}_+)^2$. If we restrict our attention to such paths then we talk about a *point-to-point polymer*, defined through the following path measure on $\Pi_{M,N}$:

$$(1.2) \quad Q_{M,N}^{\text{point}}(\pi) = \frac{1}{Z^{\text{point}}(M, N)} W_{M+N}(\pi)$$

The normalizing constant

$$Z^{\text{point}}(M, N) = \sum_{\pi \in \Pi_{M,N}} W_{M+N}(\pi)$$

is known as the *point-to-point partition function*. Similarly, if we consider all possible paths of length $2N$ then we talk about a *point-to-line polymer*, defined through the following path measure on $\bigcup_{k=-N, \dots, N} \Pi_{N+k, N-k}$ (that is, all paths of length $2N$):

$$(1.3) \quad Q_N^{\text{line}}(\pi) = \frac{1}{Z^{\text{line}}(N)} W_{2N}(\pi),$$

with the *point-to-line partition function*

$$Z^{\text{line}}(N) = \sum_{k=-N}^N Z^{\text{point}}(N+k, N-k).$$

A main quantity of interest in each case is the *free energy*, defined as the logarithm of the partition function. In the point-to-line case, another important quantity of interest is the position of the endpoint of the randomly chosen path, which we will denote by κ_N . It is widely believed that these quantities should satisfy the scalings

$$(1.4) \quad \log(Z^{\text{point}}(N, N)) \sim a_2 N + b_2 N^\chi \zeta_2,$$

$$(1.5) \quad \log(Z^{\text{line}}(N)) \sim a_1 N + b_1 N^\chi \zeta_1,$$

$$(1.6) \quad \kappa_N \sim N^\xi \mathcal{T}$$

as $N \rightarrow \infty$, where the constants a_1, a_2 and b_1, b_2 may depend on the distribution of the $\omega_{i,j}$ and β , but ζ_1, ζ_2 and \mathcal{T} should be universal up to some fairly generic assumptions on the $\omega_{i,j}$'s, while the fluctuation exponent

$$\chi = 1/3$$

and wandering exponent

$$\xi = 2/3.$$

Here, and in the rest of this article, whenever we write a relation like

$$Z_N \sim aN + bN^\kappa \zeta$$

as $N \rightarrow \infty$, what we mean is that

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\frac{Z_N - aN}{bN^\kappa} \leq m\right) = \mathbb{P}(\zeta \leq m).$$

One can also have higher, $d + 1$ dimensional versions of the model, with the paths directed in one dimension, and wandering in the other d . In all dimensions the scaling exponents χ and ξ are conjectured to satisfy the *KPZ scaling relation*

$$(1.7) \quad \chi = 2\xi - 1,$$

while the universality of the limiting distributions is unclear except in $d = 1$. For recent progress on (1.7), see [Cha12; AD11; AD12].

Although there are few results available in the general case described above, the zero-temperature limit $\beta \rightarrow \infty$, known as *last passage percolation*, is very well understood, at least for some specific choices of the environment variables $\omega_{i,j}$. Before introducing this model, we will briefly introduce the Tracy-Widom distributions from random matrix theory, which will, somewhat surprisingly, play an important role in the sequel.

1.2.2. Tracy-Widom distributions. We will restrict our attention to the distributions arising from the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE), although these are by no means the only distributions coming from random matrix theory which appear in the study of models in the KPZ universality class. The reader can consult [Meh91; AGZ10] for good expositions on random matrix theory.

We start with the unitary case. Let $\mathcal{N}(a, b)$ denote a Gaussian random variable with mean a and variance b . An $N \times N$ *GUE matrix* is an (complex-valued) Hermitian matrix A such that $A_{i,j} = \mathcal{N}(0, N/\sqrt{2}) + i\mathcal{N}(0, N/\sqrt{2})$ for $i > j$ and $A_{i,i} = \mathcal{N}(0, N)$. Here we assume that all the Gaussian variables appearing in the different entries are independent (subject to the Hermitian condition). The variance normalization by N was chosen here to make the connection with models in the KPZ class more transparent. An alternative way to describe the Gaussian Unitary Ensemble is as the probability measure on the space of $N \times N$ Hermitian matrices A with density (with respect to the Lebesgue measure on the N^2 independent parameters corresponding to the real entries on the diagonal and the real and imaginary components of the entries above the diagonal)

$$\frac{1}{Z_N} e^{-\frac{1}{2N} \text{tr} A^2}$$

for some normalization constant Z_N . If $\lambda_1^N, \dots, \lambda_N^N$ are the eigenvalues of such a matrix, then the Wigner semicircle law states that the empirical eigenvalue density $N^{-1} \sum_{i=1}^N \delta_{\lambda_i^N}$ has approximately a semicircle distribution on the interval $[-2N, 2N]$. The *Tracy-Widom GUE distribution* [TW94] arises from studying the fluctuations of the eigenvalues of a GUE matrix at the edge of the spectrum: if we denote by $\lambda_{\text{GUE}}^{\max}(N)$ the largest eigenvalue of an $N \times N$ GUE matrix then [TW94]

$$\lambda_{\text{GUE}}^{\max}(N) \sim 2N + N^{1/3} \zeta_2$$

as $N \rightarrow \infty$, where ζ_2 has the GUE Tracy-Widom distribution, which is defined as follows:

$$(1.8) \quad F_{\text{GUE}}(s) := \mathbb{P}(\zeta_2 \leq s) = \det(I - P_s K_{\text{Ai}} P_s)_{L^2(\mathbb{R})},$$

where K_{Ai} is the *Airy kernel*

$$(1.9) \quad K_{\text{Ai}}(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda),$$

$\text{Ai}(\cdot)$ is the Airy function, P_a denotes the projection onto the interval (a, ∞) , and the determinant means the Fredholm determinant on the Hilbert space $L^2(\mathbb{R})$. We will talk at length about the Airy kernel and related operators in later sections, so for now we will postpone the discussion. Fredholm determinants can be regarded as the natural generalization of the usual determinant to operators on infinite dimensional spaces. We will review their definition and properties in Section 2. Since these determinants will appear often during the rest of this introduction, the reader who is not familiar with them may want to read Section 2 before continuing.

Before continuing to the F_{GOE} we quickly note that one of the key contributions of Tracy and Widom [TW94] was to connect (1.8) to integrable systems. Let $q(s)$ be the Hastings-McLeod solution of the Painlevé II equation

$$(1.10) \quad q''(s) = 2q(s)^3 + sq(s),$$

defined by the additional boundary condition

$$(1.11) \quad q(s) \sim \text{Ai}(s) \quad \text{as } s \rightarrow \infty.$$

Then

$$F_{\text{GUE}}(s) = e^{-\int_s^\infty dx (x-s)^2 q^2(x)}.$$

The story for the Gaussian Orthogonal Ensemble is similar. An $N \times N$ *GOE matrix* is a (real-valued) symmetric matrix A such that $A_{i,j} = \mathcal{N}(0, N)$ for $i > j$ and $A_{i,i} = \mathcal{N}(0, \sqrt{2}N)$, where as before we assume that all the Gaussian variables appearing in the different entries are independent (subject to the symmetry condition). Analogously to the GUE case, the Gaussian Orthogonal Ensemble can be regarded as the probability measure on the space of $N \times N$ real symmetric matrices A with density

$$\frac{1}{Z_N} e^{-\frac{1}{4N} \text{tr } A^2}$$

for some normalization constant Z_N . As for the GUE, the Wigner semicircle law states that the empirical eigenvalue density for the GOE has approximately a semicircle distribution on the interval $[-2N, 2N]$. The fluctuations of the spectrum at its edge now give rise to the *Tracy-Widom GOE distribution*: we denote by $\lambda_{\text{GOE}}^{\max}(N)$ the largest eigenvalue of an $N \times N$ GOE matrix, then [TW96]

$$\lambda_{\text{GOE}}^{\max}(N) \sim 2N + N^{1/3} \zeta_1$$

as $N \rightarrow \infty$, where ζ_1 has the GOE Tracy-Widom distribution, defined as

$$(1.12) \quad F_{\text{GOE}}(s) := \mathbb{P}(\zeta_1 \leq s) = \det(I - P_0 B_s P_0)_{L^2(\mathbb{R})},$$

where B_s is the kernel

$$(1.13) \quad B_s(x, y) = \text{Ai}(x + y + s).$$

This Fredholm determinant formula for F_{GOE} is essentially due to [Sas05], and was proved in [FS05]. The original formula derived by Tracy and Widom is

$$(1.14) \quad F_{\text{GOE}}(s) = e^{-\frac{1}{2} \int_s^\infty dx q(x)} \sqrt{F_{\text{GUE}}(s)}$$

with q as above.

1.2.3. Last passage percolation. We come back now to our discussion about directed random polymers, and in particular their zero-temperature limit. We will restrict the discussion to *geometric last passage percolation (LPP)*, where one considers a family $\{\omega_{i,j}\}_{i,j>0}$ of independent geometric random variables with parameter q (i.e. $\mathbb{P}(\omega_{i,j} = k) = q(1-q)^k$ for $k \geq 0$). For convenience we also set for now $\omega_{i,j} = 0$ if i or j is 0. As $\beta \rightarrow \infty$, the random path measures in (1.2) and (1.3) assign an increasingly larger mass to the path π of length $K > 0$ which maximizes the weight $W_K(\pi)$. In the limit, the path measures $Q_{M,N}^{\text{point}}$ and Q_N^{line} concentrate on the maximizing path, and the quantities which play the role of the free energy are the *point-to-point last passage time*,

$$L^{\text{point}}(M, N) = \max_{\pi \in \Pi_{M,N}} \sum_{i=0}^{M+N} \omega_{\pi_i}$$

and the *point-to-line last passage time* by

$$(1.15) \quad L^{\text{line}}(N) = \max_{k=-N, \dots, N} L^{\text{point}}(N+k, N-k).$$

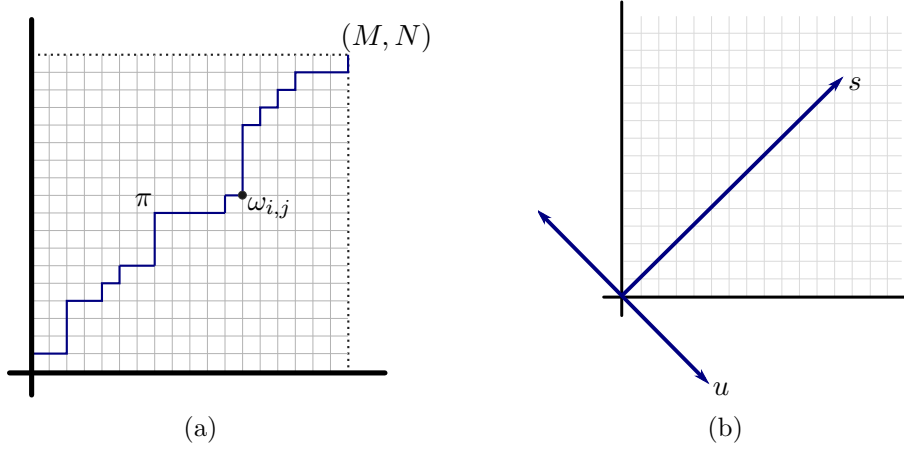


FIGURE 1. (a) A polymer/LPP path π connecting the origin to (M, N) . (b) Time s and space u axes in LPP.

Observe that these last passage times are random, as they depend on the random environment defined by the $\omega_{i,j}$.

The breakthrough, which in a sense got the whole field started, was the surprising 1999 result by Baik, Deift, and Johansson [BDJ99] which proved that the asymptotic fluctuations of the longest increasing subsequence of a random permutation have the Tracy-Widom GUE distribution. There is an intimate (and simple) connection between this model and LPP which we will not discuss, instead we will state the companion result by Johansson [Joh00] for the point-to-point LPP case:

$$(1.16) \quad L^{\text{point}}(N, N) \sim c_1 N + c_2 N^{1/3} \zeta_2,$$

where c_1, c_2 are some explicit constants which depend only on q and can be found in [Joh00] and ζ_2 has the Tracy-Widom GUE distribution. A similar result holds for point-to-line LPP. The longest increasing subsequence version goes back to Baik and Rains [BR01], while the analogue for LPP which we state here was first proved in [BFPS07] (see also [Sas05; BFP07]):

$$(1.17) \quad L^{\text{line}}(N) \sim c'_1 N + c'_2 N^{1/3} \zeta_1,$$

where ζ_1 now has the GOE Tracy-Widom distribution.

The reason why these exact results (and others we will discuss below) can be obtained for geometric last passage percolation and other related models is that LPP has an extremely rich algebraic structure which allows one to write explicit formulas for the distribution of the last passage times. The algebraic structure arises from regarding the model as a randomly growing Young tableau, where the cell (i, j) is added at time $L^{\text{point}}(i, j)$. This shift of perspective relates the problem to the representation theory of the symmetric group, and in particular to the Robinson-Schensted-Knuth (RSK) correspondence, which is the main combinatorial tool used in [Joh00] to prove the following remarkable formula:

$$\mathbb{P}(L^{\text{point}}(M, N) \leq s) = \det(I - P_s K_N^{\text{Meix}} P_s)_{L^2(\mathbb{R})}$$

for $M \leq N$, where the *Meixner kernel* K_N^{Meix} is given by

$$K_N^{\text{Meix}}(x, y) = \frac{\kappa_N}{\kappa_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y} \sqrt{w(x)w(y)},$$

$w(x) = \binom{M-N+x}{x}$, and the functions $p_N(x)$ are the normalized Meixner polynomials, i.e., the normalized family of discrete orthogonal polynomials $p_N(x)$ with respect to the weight $w(x)$, with $p_N(x)$ of degree N and leading coefficient κ_N . A non-trivial asymptotic analysis of this kernel allowed Johansson to deduce that the above Fredholm determinant converges

as $N \rightarrow \infty$ to the Fredholm determinant appearing in the definition (1.8) of the Tracy-Widom GUE distribution. A more detailed discussion of these facts is beyond the scope of this review; what the reader should keep from this discussion is that the exact results which we are discussing depend crucially on what is usually referred to as *exact solvability* or *integrability*: the availability of (extremely non-trivial) exact formulas for quantities of interest. These formulas arise from the very rich algebraic structure present in some (but by no means all) models in the KPZ class. For a recent survey on this subject see [BG12].

As part of the general KPZ universality conjecture, one expects that (1.16) and (1.17) hold not only for LPP, but in general for any $\beta > 0$. In other words, the belief is that in (1.4) and (1.5), the random variables ζ_2 and ζ_1 have respectively the Tracy-Widom GUE and GOE distributions. There has been only partial progress in proving this conjecture for point-to-point directed polymers (and virtually none in the point-to-line case), the difficulty lying in the lack of exact solvability. Versions of this conjecture have been proved for two related models in the point-to-point case: the continuum random polymer in [ACQ11] (building on results of [TW08a; TW08b; TW09]) and the semi-discrete polymer of O'Connell and Yor in [BC11; BCF12] (see also [O'C12]). In the setting of discrete directed random polymers, [COSZ11] showed that if the weights are chosen so that $-w_{i,j}$ is distributed as the logarithm of a Gamma random variable with parameter $\theta_i + \hat{\theta}_j$ (for some fixed θ_i 's and $\hat{\theta}_j$'s) then the model is exactly solvable in the sense explained above. This was later used in [BCR13] to prove that the asymptotic fluctuations of the free energy of the point-to-point polymer (at least for low enough temperature) have the conjectured Tracy-Widom GUE distribution.

1.2.4. Spatial fluctuations and the Airy processes. The Airy processes arise from LPP when we look not only at the fluctuations of the free energy at a single site, but instead at several sites. To this end, we define the rescaled point-to-point process $u \mapsto H_N^{\text{point}}(u)$ by linearly interpolating the values given by scaling $L^{\text{point}}(M, N)$ through the relation

$$(1.18) \quad L^{\text{point}}(N + u, N - u) = c_1 N + c_2 N^{1/3} H_N^{\text{point}}(c_3 N^{-2/3} u)$$

for $u = -N, \dots, N$, where the constants c_i have explicit expressions which depend only on q and can be found in [Joh03]. Observe that this corresponds to looking at the free energy at a line of slope -1 passing through (N, N) . The limiting behavior of H_N^{point} is described by the Airy_2 process \mathcal{A}_2 (minus a parabola, see Theorem 1.1). This process was introduced by Prähofer and Spohn [PS02], and is defined through its finite-dimensional distributions, which are given by a Fredholm determinant formula: given $x_0, \dots, x_n \in \mathbb{R}$ and $u_1 < \dots < u_n$ in \mathbb{R} ,

$$(1.19) \quad \mathbb{P}(\mathcal{A}_2(u_1) \leq x_1, \dots, \mathcal{A}_2(u_n) \leq x_n) = \det(I - f^{1/2} K_{\text{Ai}}^{\text{ext}} f^{1/2})_{L^2(\{u_1, \dots, u_n\} \times \mathbb{R})},$$

where we have counting measure on $\{u_1, \dots, u_n\}$ and Lebesgue measure on \mathbb{R} , f is defined on $\{u_1, \dots, u_n\} \times \mathbb{R}$ by

$$(1.20) \quad f(u_j, x) = \mathbf{1}_{x \in (x_j, \infty)},$$

and the *extended Airy kernel* [PS02; FNH99; Mac94] is defined by

$$(1.21) \quad K_{\text{Ai}}^{\text{ext}}(u, \xi; u', \xi') = \begin{cases} \int_0^\infty d\lambda e^{-\lambda(u-u')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda), & \text{if } u \geq u' \\ -\int_{-\infty}^0 d\lambda e^{-\lambda(u-u')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda), & \text{if } u < u'. \end{cases}$$

Although it is not obvious from the definition, the Airy_2 process is stationary (this will become clear in Section 3.1), and as should be expected from (1.16), $\mathbb{P}(\mathcal{A}_2(u) \leq m) = F_{\text{GUE}}(m)$ for all u . There is a close connection, which we will explain in Section 1.5, between the Airy kernel K_{Ai} appearing in the definition (1.8) of the Tracy-Widom GUE distribution and the extended kernel $K_{\text{Ai}}^{\text{ext}}$.

The precise result linking the point-to-point LPP spatial fluctuations to the Airy_2 process is due to Johansson (see also [PS02]):

Theorem 1.1 (Johansson [Joh03]). *There is a continuous version of \mathcal{A}_2 , and*

$$H_N^{\text{point}}(u) \xrightarrow{N \rightarrow \infty} \mathcal{A}_2(u) - u^2$$

in distribution in the topology of uniform convergence of continuous functions on compact sets.

In the LPP picture, the “time” variable (which we will denote by s) flows in the $(1, 1)$ direction of the plane, while “space” (which, as above, we will denote by u) corresponds to the direction $(1, -1)$ (see Figure 1b). In this sense, Theorem 1.1 describes the spatial fluctuations of the point-to-point last passage times as time $s \rightarrow \infty$.

One can think of extending the LPP model to paths starting at $s = 0$ with any space coordinate, i.e., paths which start at any point of the form $(k, -k)$, $k \in \mathbb{Z}$. To recover point-to-point LPP one simply sets $\omega_{i,j} = 0$ whenever $i \leq 0$ or $j \leq 0$, which is easily seen to be equivalent (from the point of view of last passage times) to forcing our paths to start at the origin. In this sense, point-to-point LPP and the Airy_2 process correspond to the δ_0 (also known as *delta*, *narrow wedge* or *curved*) initial data (see Figure 2a). Note that in this case we only assign positive weights to sites such that $s > |u|$. To recover the flat, stationary and mixed initial data which we introduced earlier, we need to assign weights to sites such that $s \leq |u|$.

Remark 1.2. The results for the flat, stationary and mixed initial data have been proved in settings which differ slightly from the one introduced here. To avoid additional notation and complications, we will state the results on the present setting. We refer the reader to the corresponding references for more details on the differences. In the case of multipoint results, one can translate between the various settings by using the slow decorrelation result proved of [Fer08; CFP12], as done in [BFP10; CFP10].

We start with the *flat* initial data. It corresponds to extending the weights $\omega_{i,j}$ to be independent geometric random variables with parameter q whenever $i + j > 0$ and setting $\omega_{i,j} = 0$ otherwise (see Figure 2b). This corresponds to letting our paths start at any site in the line $s = 0$ but not attach any additional weights along that line, which explains the name flat. The corresponding point-to-line rescaled process may be defined as follows: first we extend the definition of last passage times to accomodate the flat initial data,

$$L_{\text{flat}}^{\text{point}}(M, N) = \max_{i \in \mathbb{Z}} \max_{\pi \in \Pi_{(i, -i) \rightarrow (M, N)}} \sum_{j=0}^{2i+M+N} \omega_{\pi_j}$$

with self-explanatory notation, and then we define the rescaled process $u \mapsto H_N^{\text{line}}(u)$ by linearly interpolating the values given the relation

$$L_{\text{flat}}^{\text{point}}(N + u, N - u) = c_1 N + c_2 N^{1/3} H_N^{\text{line}}(c_3 N^{-2/3} u)$$

for $u = -N, \dots, N$. The flat initial data gives rise to the Airy_1 process \mathcal{A}_1 , which was introduced by Sasamoto [Sas05], and is defined through its finite-dimensional distributions,

$$(1.22) \quad \mathbb{P}(\mathcal{A}_1(u_1) \leq \xi_1, \dots, \mathcal{A}_1(u_n) \leq \xi_n) = \det(I - \mathbf{f} K_1^{\text{ext}} \mathbf{f})_{L^2(\{u_1, \dots, u_n\} \times \mathbb{R})},$$

with \mathbf{f} as in (1.20) and

$$(1.23) \quad K_1^{\text{ext}}(u, \xi; u', \xi') = -\frac{1}{\sqrt{4\pi(u' - u)}} \exp\left(-\frac{(\xi' - \xi)^2}{4(u' - u)}\right) \mathbf{1}_{u' > u} \\ + \text{Ai}(\xi + \xi' + (u' - u)^2) \exp((u' - u)(\xi + \xi') + \frac{2}{3}(u' - u)^3).$$

The Airy_1 process is stationary, and as should be expected from (1.17), its marginals are given by the Tracy-Widom GOE distribution: $\mathbb{P}(\mathcal{A}_1(u) \leq m) = F_{\text{GOE}}(2m)$ for all u .

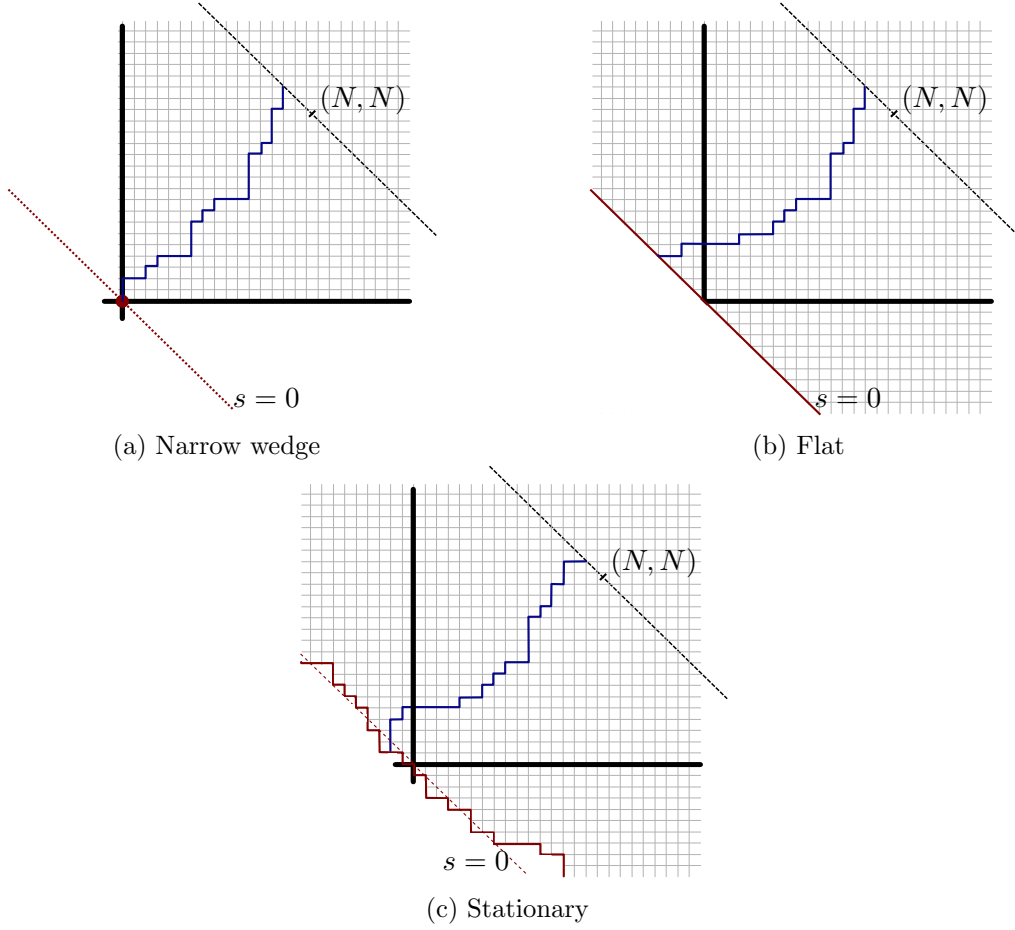


FIGURE 2. Schematic representation of the LPP models with asymptotic spatial fluctuations given by: (a) Airy_2 ; (b) Airy_1 ; (c) $\text{Airy}_{\text{stat}}$.

Theorem 1.3 ([BFPS07; BFP07; BFS08a]).

$$H_N^{\text{line}}(u) \xrightarrow{N \rightarrow \infty} 2^{1/3} \mathcal{A}_1(2^{-2/3}u)$$

in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1.2).

The powers of $2^{1/3}$ in the above limit should be regarded as an arbitrary normalization (in fact, one could have defined the Airy_1 process as this scaled version of it). The appearance of these factors has to do with the fact that the natural scaling in the definition of these quantities differs between random matrix models and models such as LPP or directed polymers (as, for instance, in (1.25) below). See Section 2 of [CQR13] for a related discussion.

The *stationary* initial data is slightly more cumbersome to introduce. The name stationary comes from the fact that for the closely related totally asymmetric exclusion process (TASEP), this initial condition corresponds to starting with particles placed according to a product Bernoulli measure with parameter $1/2$, which is stationary for the process. Translated to LPP, this initial condition corresponds to the following. Let $(S_n)_{n \in \mathbb{Z}}$ be the path of a double-sided simple random walk on \mathbb{Z} with $S_0 = 0$, which we assume to be independent of the weights $\omega_{i,j}$. We rotate this random walk path by an angle of $-\pi/4$ and then put it along the $s = 0$ line by defining the (random) discrete curve $\gamma_0 = \{(\frac{1}{2}(S(i) + i), \frac{1}{2}(S(i) - i)), i \in \mathbb{Z}\}$. We then extend the weights $\omega_{i,j}$ to be independent geometric random variables with parameter q whenever (i, j) lies above γ_0 and $\omega_{i,j} = 0$

otherwise (see Figure 2c). The corresponding stationary rescaled process $H_N^{\text{stat}}(u)$ can be defined analogously to the previous cases, by maximizing over paths starting at γ_0 and going to the anti-diagonal line passing through (N, N) . It gives rise to the $\text{Airy}_{\text{stat}}$ process $\mathcal{A}_{\text{stat}}$. Its definition is also given in terms of finite-dimensional distributions involving Fredholm determinants, but the formulas are a lot more cumbersome. We will not need the exact formulas, so we refer the reader to [BFP10] for the details. Despite its name, $\mathcal{A}_{\text{stat}}$ is not stationary as a process. In fact, due to the connection with stationary TASEP, $\mathcal{A}_{\text{stat}}$ is just a standard double-sided Brownian motion, but with a non-trivial random height shift at the origin given by the Baik-Rains distribution, see [BR00]. The convergence result in this case is the following:

Theorem 1.4 ([BFP10]).

$$H_N^{\text{stat}}(u) \xrightarrow[N \rightarrow \infty]{} \mathcal{A}_{\text{stat}}(u)$$

in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1.2).

The mixed initial conditions can be obtained by placing one condition on each half of the line $u = 0$. We will explain how this is done in the case of the *half-flat*, or *wedge*→*flat* initial data, and leave the examples leading to $\mathcal{A}_{2 \rightarrow \text{BM}}$ and $\mathcal{A}_{1 \rightarrow \text{BM}}$ to the interested reader (see [BFS09; CFP10]). To obtain the $\text{Airy}_{2 \rightarrow 1}$ process we extend the weights $\omega_{i,j}$ to be independent geometric random variables with parameter q whenever $i, j > 0$, or $i + j > 0$ with $i < 0$, setting $\omega_{i,j} = 0$ for all other sites. The half-flat rescaled process $H_N^{\text{half-line}}(u)$ is obtained as in the previous cases, and gives rise to the $\text{Airy}_{2 \rightarrow 1}$ process $\mathcal{A}_{2 \rightarrow 1}$. It was introduced by Borodin, Ferrari, and Sasamoto [BFS08b], and is given by

$$\mathbb{P}(\mathcal{A}_{2 \rightarrow 1}(u_1) \leq \xi_1, \dots, \mathcal{A}_{2 \rightarrow 1}(u_m) \leq \xi_m) = \det(I - f K_{2 \rightarrow 1}^{\text{ext}} f)_{L^2(\{u_1, \dots, u_m\} \times \mathbb{R})},$$

with f as in (1.20) and

$$(1.24) \quad K_{2 \rightarrow 1}^{\text{ext}}(u, \xi; u', \xi') = -\frac{1}{\sqrt{4\pi(\xi' - \xi)}} \exp\left(-\frac{(\tilde{\xi}' - \tilde{\xi})^2}{4(u' - u)}\right) \mathbf{1}_{u' > u} \\ + \frac{1}{(2\pi i)^2} \int_{\gamma_+} dw \int_{\gamma_-} dz \frac{e^{w^3/3 + u'w^2 - \tilde{\xi}'w}}{e^{z^3/3 + uz^2 - \tilde{\xi}z}} \frac{2w}{(z - w)(z + w)},$$

where $\tilde{\xi} = \xi - u^2 \mathbf{1}_{u \leq 0}$, $\tilde{\xi}' = \xi' - (u')^2 \mathbf{1}_{u' \leq 0}$ and the paths γ_+, γ_- satisfy $-\gamma_+ \subseteq \gamma_-$ with $\gamma_+ : e^{i\phi_+}\infty \rightarrow e^{-i\phi_+}\infty$, $\gamma_- : e^{-i\phi_-}\infty \rightarrow e^{i\phi_-}\infty$ for some $\phi_+ \in (\pi/3, \pi/2)$, $\phi_- \in (\pi/2, \pi - \phi_+)$. As could be expected from the above description, the $\text{Airy}_{2 \rightarrow 1}$ process crosses over between the Airy_2 and the Airy_1 processes in the sense that $\mathcal{A}_{2 \rightarrow 1}(u + v)$ converges to $2^{1/3} \mathcal{A}_1(2^{-2/3}u)$ as $v \rightarrow \infty$ and $\mathcal{A}_2(u)$ when $v \rightarrow -\infty$. The convergence result is the following:

Theorem 1.5 ([BFS08b]).

$$H_N^{\text{half-line}}(u) - u^2 \mathbf{1}_{u \leq 0} \xrightarrow[N \rightarrow \infty]{} \mathcal{A}_{2 \rightarrow 1}(u)$$

in the sense of convergence of finite-dimensional distributions (on a slightly different setting than the one presented here, see Remark 1.2).

Some of these results have been extended to the case where the points at which one computes the corresponding finite-dimensional distributions do not all lie in the same anti-diagonal line, but instead fall in certain *space-like curves* lying close enough to such a line, see e.g. [BF08; BFS08a] for more details and [CFP10] for further extensions.

From the definitions it is clear that the basic three Airy processes \mathcal{A}_2 , \mathcal{A}_1 , and $\mathcal{A}_{\text{stat}}$ are invariant under $\mathcal{A}(u) \mapsto \mathcal{A}(-u)$, but the mixed cases are not.

Since all initial data are superpositions of Dirac masses, there is a sense in which the Airy₂ process is the most basic. For example, using the fact that point-to-line last passage times are computed simply as the maximum of point-to-point last passage times, Johansson [Joh03] obtained the following celebrated formula as a corollary of (1.16) and Theorem 1.1:

$$(1.25) \quad \mathbb{P}\left(\sup_{x \in \mathbb{R}} \{\mathcal{A}_2(x) - x^2\} \leq m\right) = F_{\text{GOE}}(4^{1/3}m).$$

A direct proof of this formula was later provided in [CQR13]. The argument used in this second proof starts with a different expression for the finite-dimensional distributions of \mathcal{A}_2 in terms of the Fredholm determinant of a certain boundary value operator. This type of formula, and their extensions to continuum statistics, are the starting point of most of the results we will survey in Sections 3.2 and 4. For example, as described in Section 4.2, they allow to compute the asymptotic distribution of κ_N , the position of the endpoint in the maximizing path in point-to-line LPP.

Extrapolating from (1.25) leads to a conjecture that the one-point marginals of the other Airy processes should be obtained through certain variational problems involving the Airy₂ process. To state the precise conjectures we turn to the stochastic heat equation, whose logarithm is the solution of the KPZ equation. The advantage of this model over LPP and other discrete models is that it is linear in the initial data, and hence the heuristics are more easily stated in that context. The disadvantage is that most of the argument relies on conjectures based on universality.

1.3. The continuum random polymer and the stochastic heat equation. We now consider the continuum version of the finite temperature discrete random polymers (1.2) and (1.3). The (point-to-point) continuum random polymer is a random probability measure $P_{T,x}^{\beta,\xi}$ on continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$ and $x(T) = x$ with formal weight

$$e^{-\beta \int_0^T dt \xi(t, x(t)) - \frac{1}{2} \int_0^T dt |\dot{x}(t)|^2}$$

given to the path $x(\cdot)$, where $\xi(t, x)$, $t \geq 0$, $x \in \mathbb{R}$ is space-time white noise, i.e. the distribution-valued Gaussian variable such that for smooth functions φ of compact support in $\mathbb{R}_+ \times \mathbb{R}$, $\langle \varphi, \xi \rangle := \int_{\mathbb{R}_+ \times \mathbb{R}} dt dx \varphi(t, x) \xi(t, x)$ are mean zero Gaussian random variables with covariance structure $E[\langle \varphi_1, \xi \rangle \langle \varphi_2, \xi \rangle] = \langle \varphi_1, \varphi_2 \rangle$. One can also think of the continuum random polymer as having a density

$$e^{-\beta \int_0^T dt \xi(t, x(t))}$$

with respect to the Brownian bridge. Neither prescription makes mathematical sense, but the second one does if one smooths out the white noise $\xi(t, x)$ in space. Removing the smoothing, one find that there is indeed a limiting measure supported on continuous functions $C[0, T]$ which we call $P_{T,x}^{\beta,\xi}$. In fact, it is a Markov process, and one can define it directly as follows. Let $z(s, x, t, y)$ denote the solution of the stochastic heat equation after time $s \geq 0$ starting with a delta function at x ,

$$(1.26) \quad \partial_t z = \frac{1}{2} \partial_y^2 z - \beta \xi z, \quad t > s, \quad y \in \mathbb{R}, \quad z(s, x, s, y) = \delta_x(y).$$

It is important that they are all using the same noise ξ . Note that the stochastic heat equation is well-posed [Wal86]. The solutions look locally like exponential Brownian motion in space. They are Hölder $\frac{1}{2} - \delta$ for any $\delta > 0$ in x and $\frac{1}{4} - \delta$ for any $\delta > 0$ in t . In fact, exponential Brownian motion $e^{B(x)}$ is invariant up to multiplicative constants, i.e. if one starts (1.26) with $e^{B(x)}$ where $B(x)$ is a two-sided Brownian motion, then there is a (random) $C(t)$ so that $C(t)z(t, x)$ is an exponential of another two-sided Brownian motion [BG97]. $P_{T,x}^{\beta,\xi}$ is then defined to be the probability measure on continuous functions $x(t)$ on

$[0, T]$ with $x(0) = 0$ and $x(T) = x$ and finite dimensional distributions

$$\begin{aligned} P_{T,x}^{\beta,\xi}(x(t_1) \in dx_1, \dots, x(t_n) \in dx_n) \\ = \frac{z(0, 0, t_1, x_1)z(t_1, x_1, t_2, x_2) \cdots z(t_{n-1}, x_{n-1}, t_n, x_n)z(t_n, x_n, T, x)}{z(0, 0, T, x)} dx_1 \cdots dx_n \end{aligned}$$

for $0 < t_1 < t_2 < \cdots < t_n < T$. One can check these are a.s. a consistent family of finite dimensional distributions. This holds basically because of the Chapman-Kolmogorov equation

$$\int_{-\infty}^{\infty} du z(s, x, \tau, u)z(\tau, u, t, y) = z(s, x, t, y)$$

for $s < \tau \leq t$, which is a consequence of the linearity of the stochastic heat equation.

Note that the construction is for each $T > 0$ fixed. Unlike the usual case of diffusions, the measures are very inconsistent for varying T . One should imagine that the polymer paths are peeking into the future to see the best route, so the measure depends considerably on all the noise in the time interval $[0, T]$. We can also define the joint measure $\mathbb{P}_{T,x}^{\beta,\xi} = P_{T,x}^{\beta,\xi} \otimes Q(\xi)$ where Q is the distribution of the ξ , i.e. the probability measure of the white noise.

Theorem 1.6 ([AKQ12b]).

- (i) The measures $P_{T,x}^{\beta,\xi}$ and $\mathbb{P}_{T,x}^{\beta}$ are well-defined (the former, Q -almost surely).
- (ii) $P_{T,x}^{\beta,\xi}$ is a Markov process supported on Hölder continuous functions of exponent $\frac{1}{2} - \delta$ for any $\delta > 0$, for Q -almost every ξ .
- (iii) Let $t_k^n = \frac{k}{2^n}$. Then with $\mathbb{P}_{T,x}^{\beta}$ probability one, we have that for all $0 \leq t \leq 1$

$$\sum_{k=1}^{\lfloor 2^n t \rfloor} (x(t_k^n) - x(t_{k-1}^n))^2 \xrightarrow{n \rightarrow \infty} t,$$

i.e. the quadratic variation exists, and coincides with the one obtained for $\mathbb{P}_{T,x}^0$ (the Brownian bridge measure).

- (iv) $P_{T,x}^{\beta,\xi}$ is singular with respect to $\mathbb{P}_{T,x}^0$ (the Brownian bridge measure) for Q -almost every ξ .

So the continuum random polymer looks locally like, but is singular with respect to, Brownian motion. One can also define the point-to-line continuum random polymer \mathbb{P}_T^{β} , in the same way as in the discrete case. For large T , one expects $\text{Var}_{\mathbb{P}_T^{\beta}}(x(T)) \sim T^{4/3}$ in the point-to-line case or $\text{Var}_{\mathbb{P}_{T,0}^{\beta}}(x(T/2)) \sim T^{4/3}$ in the point-to-point case. Here the variance is over the random background as well as $P_{T,x}^{\beta,\xi}$. The conditional variance given ξ should be much smaller.

If $z(t, x)$ is the solution of (1.26) then $h(t, x) = -\beta^{-1} \log z(t, x)$ can be thought of as either the (renormalized) free energy of the point-to-point continuum random polymer, or the Hopf-Cole solution of the Kardar-Parisi-Zhang equation,

$$(1.27) \quad \partial_t h = -\frac{\beta}{2} (\partial_x h)^2 + \frac{1}{2} \partial_x^2 h + \xi,$$

for random interface growth. Since $\log z(t, x)$ looks locally like Brownian motion, (1.27) is not well-posed (see [Hai13] for recent progress on this question.) If ξ were smooth, then the Hopf-Cole transformation takes (1.26) to (1.27). For white noise ξ , we take $h(t, x) = -\beta^{-1} \log z(t, x)$ with $z(t, x)$ a solution of (1.26) to be the *definition* of the solution of (1.27). It is known [BG97] that these are the solutions one obtains if one smooths the noise, solves the equation, and takes a limit as the smoothing is removed (and after subtraction of a diverging constant). They are also the solutions obtained as the limit

of discrete models like asymmetric exclusion in the weakly asymmetric limit [BG97], or directed polymers in the *intermediate disorder limit* [AKQ12a].

To understand the intermediate disorder limit we consider how the KPZ equation (1.27) rescales. Let

$$h_\epsilon(t, x) = \epsilon^a h(\epsilon^{-z}t, \epsilon^{-1}x)$$

Recall the white noise has the distributional scale invariance

$$\xi(t, x) \stackrel{\text{dist}}{=} \epsilon^{\frac{z+1}{2}} \xi(\epsilon^z t, \epsilon^1 x).$$

Hence, setting $\beta = 1$ for clarity,

$$\partial_t h_\epsilon = -\frac{1}{2}\epsilon^{2-z-a}(\partial_x h_\epsilon)^2 + \frac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{a-\frac{1}{2}z+\frac{1}{2}}\xi.$$

Because the paths of h are locally Brownian in x we are forced to take $a = 1/2$ to see non-trivial limiting behaviour. This forces us to take

$$z = 3/2$$

The non-trivial limiting behaviour of models in the KPZ universality class are all obtained in this scale.

On the other hand, if we started with KPZ with noise of order $\epsilon^{1/2}$,

$$\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \epsilon^{1/2}\xi,$$

then a diffusive scaling,

$$h_\epsilon(t, x) = h(\epsilon^{-2}t, \epsilon^{-1}x),$$

would bring us back to the standard KPZ equation (1.27). This is the intermediate disorder scaling in which KPZ and the continuum random polymer can be obtained from discrete directed polymers. It tells us that if we set

$$\beta = \epsilon^{1/2}\tilde{\beta}$$

in (1.1) then the distribution of the rescaled polymer path

$$x_\epsilon(t) := \epsilon x_{\lfloor \epsilon^{-2}t \rfloor} \quad 0 \leq t \leq T$$

will converge to the continuum random polymer, with temperature $c\tilde{\beta}$ (see [AKQ12a] for details).

1.4. General conjectural picture for the SHE. Define A_t from the solution of (1.26) by

$$(1.28) \quad z(0, y; t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))}.$$

$A_t(\cdot)$ is called the *crossover Airy process*, the key conjecture being

$$(1.29) \quad A_t(x) \rightarrow \mathcal{A}_2(x)$$

This is known in the sense of one-dimensional distributions (see [ACQ11], where (1.29) is Conjecture 1.5). A non-rigorous derivation based on a factorization approximation for the Bethe eigenfunctions of the δ -Bose gas can be found in [PS11]. Note however that the factorization assumption is almost certainly false.

Now one tries to use the linearity of the stochastic heat equation to solve for general initial data $z(0, x) = z_0(x)$,

$$(1.30) \quad z(t, x) = \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))} z_0(y).$$

It is not hard to see that the equality is correct in the sense of one-dimensional distributions, but not more. If one wants, for example, joint distributions of $z(t, x_i)$ for more than one x_i ,

then one needs to enhance the crossover Airy process in (1.28) to a two parameter process $A_t(2^{-1/3}t^{-2/3}x, 2^{-1/3}t^{-2/3}y)$. The conjectural limit of this is a two parameter process we call the *Airy sheet*. However, we do not even have a full conjecture for its finite dimensional distributions, though some properties can be described (see [CQ11]).

Calling $\tilde{x} = 2^{-1/3}t^{-2/3}x$ and $\tilde{y} = 2^{-1/3}t^{-2/3}y$ and starting with initial data $z_0(x) = \exp\{2^{-1/3}t^{1/3}f(2^{-1/3}t^{-2/3}x)\}$, we can rewrite the exponent in (1.30) as

$$2^{-1/3}t^{1/3}[A_t(\tilde{x} - \tilde{y}) - (\tilde{x} - \tilde{y})^2 - f(\tilde{y})] - \frac{1}{24}t$$

so that for large t the fluctuation field $2^{1/3}t^{-1/3}[\log z(t, x) + \frac{1}{24}t + \log(\sqrt{2\pi t})]$ is well approximated by

$$\sup_{\tilde{y} \in \mathbb{R}} \{\mathcal{A}_2(\tilde{x} - \tilde{y}) - (\tilde{x} - \tilde{y})^2 - \tilde{f}(\tilde{y})\}.$$

The type of initial data would appear to be quite restrictive, but actually this picks out the appropriate self-similar classes. The easiest example is the flat case $f = 0$. We obtain the statement

$$(1.31) \quad \mathcal{A}_1(x) \stackrel{(d)}{=} \sup_{y \in \mathbb{R}} \{\mathcal{A}_2(y - x) - (y - x)^2\}$$

in the sense of one-dimensional distributions. Since the left hand side is just the GOE Tracy-Widom law this is the well known theorem of Johansson (1.25) once again.

If one starts with a two sided Brownian motion, then the required self-similarity of this initial data is just the Brownian scaling and one arrives at¹

$$\mathcal{A}_{\text{stat}}(x) \stackrel{(d)}{=} \sup_{y \in \mathbb{R}} \{\mathcal{A}_2(y - x) - (y - x)^2 - \sqrt{2}B(y)\}.$$

The mixed cases require a tiny bit more care. Let's explain the heuristics first for the case of the $\text{Airy}_{2 \rightarrow 1}$ process. Starting from the step initial data $z(0, x) = \mathbf{1}_{x > 0}$ the prediction is

$$(1.32) \quad -\log z(t, x) \approx \frac{1}{2t}x^2\mathbf{1}_{x < 0} + \frac{1}{24}t + \log(\sqrt{2\pi t}) - 2^{-1/3}t^{1/3}\mathcal{A}_{2 \rightarrow 1}(2^{-1/3}t^{-2/3}x).$$

On the other hand, by linearity we have for each fixed x , in distribution,

$$(1.33) \quad z(t, x) = \int_0^\infty dy z(0, y; t, x) = \int_0^\infty dy \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))}.$$

Comparing with (1.32) we deduce that the processes $\sup_{y \geq 0} (\mathcal{A}_2(x - y) - (x - y)^2)$ and $\mathcal{A}_{2 \rightarrow 1}(x) - x^2\mathbf{1}_{x < 0}$ should have the same one-dimensional distribution or, equivalently, that

$$(1.34) \quad \mathcal{A}_{2 \rightarrow 1}(x) - x^2\mathbf{1}_{x < 0} \stackrel{(d)}{=} \sup_{y \leq x} \{\mathcal{A}_2(y) - y^2\}$$

for each fixed $x \in \mathbb{R}$. This distributional identity has actually been proved rigourously, and its proof is based on the methods we will survey in Sections 3 and 4 (see Theorem 4.6).

The same heuristic argument works for the other two crossover cases. If we let $z(0, x) = e^{B(x)}\mathbf{1}_{x \geq 0}$, where $B(x)$ is a standard Brownian motion, then (1.32) and (1.33) are replaced respectively by

$$-\log z(t, x) \approx \frac{1}{2t}x^2\mathbf{1}_{x < 0} + \frac{1}{24}t + \log(\sqrt{2\pi t}) - 2^{-1/3}t^{1/3}\mathcal{A}_{2 \rightarrow \text{BM}}(2^{-1/3}t^{-2/3}x)$$

and

$$z(t, x) = \int_0^\infty dy z(0, y; t, x) = \int_0^\infty dy \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t} - \frac{t}{24} + B(y) + 2^{-1/3}t^{1/3}A_t(2^{-1/3}t^{-2/3}(x-y))},$$

¹We thank J. Baik and Z. Liu for pointing out the missing $\sqrt{2}$ on the right hand side of this equality in an earlier version of this manuscript. See [BL13a] for more details.

and now the same scaling argument allows to conjecture that

$$\mathcal{A}_{2 \rightarrow \text{BM}}(x) - x^2 \mathbf{1}_{x < 0} \stackrel{(d)}{=} \sup_{y \leq x} (\mathcal{A}_2(y) + \tilde{B}(x - y) - y^2)$$

for each fixed $x \in \mathbb{R}$, where now $\tilde{B}(y)$ is a Brownian motion with diffusion coefficient 2. An analogous argument with $z(0, x) = \mathbf{1}_{x \leq 0} + e^{B(x)} \mathbf{1}_{x \geq 0}$ translates into conjecturing that

$$\mathcal{A}_{1 \rightarrow \text{BM}}(x) \stackrel{(d)}{=} \sup_{y \in \mathbb{R}} (\mathcal{A}_2(y) + \tilde{B}(x - y) \mathbf{1}_{y \leq x} - y^2)$$

for each fixed $x \in \mathbb{R}$. As we explained, these equalities in distribution will only hold in the sense of one-dimensional distributions, i.e. for each fixed x .

The strategy used in the proof of (1.34) is considerably more difficult to implement for the other two crossover cases (see Section 4.4 and the discussion at the end of Section 1.2 in [QR13]), and in fact these identities remain conjectures for now. In work in progress [CLW] obtain an improved version of the slow decorrelation result proved in [CFP12], which should allow to prove a general version of formulas for last passage times in last passage percolation in terms of variational problems for the Airy_2 process. In particular, such a result would give a proof of these conjectural formulas.

1.5. Determinantal formulas and extended kernels. As we already mentioned, the results we will survey in Sections 3 and 4 are based on alternative Fredholm determinant formulas for the finite-dimensional distributions of the Airy processes. We will introduce these formulas in Section 3, but before doing that let us explain why the original extended kernel formulas are natural. We will first do this in a simpler setting, namely random point processes on finite sets. Then we will explain how similar arguments can be used to derive the formula (1.19) for the finite-dimensional distributions of the Airy_2 process.

1.5.1. Extended kernels and the Eynard-Mehta Theorem. Let \mathcal{X} be a finite set. A *random point process* on \mathcal{X} is a probability measure on the family $2^{\mathcal{X}}$ of subsets of \mathcal{X} , which we think of as point configurations. A random point process is called *determinantal* if there exists a $|\mathcal{X}| \times |\mathcal{X}|$ matrix K with rows and columns indexed by the elements of \mathcal{X} such that

$$\rho(A) := \mathbb{P}(\{X \in 2^{\mathcal{X}} : A \subseteq X\}) = \det(K|_A),$$

where \mathbb{P} is the probability measure underlying the point process and $K|_A$ is the submatrix of K indexed by A ,

$$K_A = [K(x, x')]_{x, x' \in A}.$$

The function ρ is called the *correlation function* of the process, and K is called its *correlation kernel*. For more details see [Bor11] and references therein. The term determinantal was introduced in [BO00].

We are interested in a particular type of random point processes. Let $\mathcal{X}^1, \dots, \mathcal{X}^n$ be n disjoint finite sets. We consider a point process supported on kn -point configurations with the property that there are exactly k points in each \mathcal{X}^i . The probability of such a configuration is given as follows: given collections of points $\{x_i^j\}_{i=1, \dots, k} \subseteq \mathcal{X}^j$ for $j = 1, \dots, n$, we set

$$(1.35) \quad \mathbb{P}(\{\{x_i^1\}_{i=1, \dots, k} \cup \dots \cup \{x_i^n\}_{i=1, \dots, k}\}) \\ = Z^{-1} \det [\phi_i(x_j^1)]_{i, j=1}^k \det [W_1(x_i^1, x_j^2)]_{i, j=1}^k \cdots \det [W_{n-1}(x_i^{n-1}, x_j^n)]_{i, j=1}^k \det [\psi_i(x_j^n)]_{i, j=1}^k,$$

where the ϕ_i 's are some functions on \mathcal{X}^1 , the ψ_i 's are some functions on \mathcal{X}^n and the W_i 's are matrices with rows indexed by \mathcal{X}^i and columns indexed by \mathcal{X}^{i+1} . The normalization constant Z is chosen so that the total mass of the measure is 1. We are assuming implicitly that the right hand side above is non-negative for any admissible point configuration.

Write Φ for the $k \times |\mathcal{X}^1|$ matrix with k rows and columns indexed by elements of \mathcal{X}^1 which is defined by $\Phi_{i,x} = \phi_i(x)$ for $1 \leq i \leq k$ and $x \in \mathcal{X}^1$. Similarly, write Ψ for the $|\mathcal{X}^n| \times k$ matrix with k columns and rows indexed by \mathcal{X}^k which is defined by $\Psi_{x,i} = \psi_i(x)$ for $1 \leq i \leq k$ and $x \in \mathcal{X}^n$. Furthermore, define the $k \times k$ matrix

$$M = \Phi W_1 \cdots W_{n-1} \Psi.$$

We will assume that $\det(M) \neq 0$. Under this assumption it can be shown (see e.g. [BR05]) that the normalization constant Z in (1.35) equals $\det(M)$.

The Eynard-Mehta Theorem states that a random point processes defined as in (1.35) is determinantal. Moreover, the theorem gives an explicit formula for the correlation kernel. The precise statement is the following:

Theorem 1.7 ([EM98]). *The random point processes defined by (1.35) is determinantal. Its correlation kernel is the block matrix K with $n \times n$ blocks, such that the (i, j) -block has rows indexed by \mathcal{X}^i and columns indexed by \mathcal{X}^j , and is given by*

$$K_{i,j} = W_i \cdots W_{n-1} \Psi M^{-1} \Phi W_1 \cdots W_{j-1} - W_i \cdots W_{j-1}.$$

For a simple proof of this result see [BR05]. Remarkably, the inverse M^{-1} can be computed, or at least approximated, in many cases of interest.

The connection with the models we have discussed so far is through certain families of non-intersecting paths. The Airy₂ process can be obtained directly as a limit of the top line of several different families of non-intersecting paths, one of which is presented in Section 1.5.2 (for some others see [Joh06]). For the other Airy processes presented in Section 1.2.4 the connection with non-intersecting paths is less immediate (see for instance the discussion preceding Lemma 3.4 in [BFPS07]), but in any case enough of the above structure remains, and the proofs still rely crucially on a version of Theorem 1.7. On the other hand, we may think of the kn -point configurations where the measure defined in (1.35) is supported as defining a family of k (in principle not necessarily non-intersecting) paths. For example, the first path would be expressed by $(x_1^1, x_1^2, \dots, x_1^n)$. It turns out, as we will see below, that probability measures on families of k non-intersecting paths on $\mathcal{X}^1 \cup \dots \cup \mathcal{X}^n$ are naturally given by expressions like (1.35), and hence have a determinantal structure. If the sets \mathcal{X}^i are endowed with some total order and we assume that our non-intersecting paths are arranged so that $(x_1^1, x_1^2, \dots, x_1^n)$ is the top path, then one can prove (see e.g. [Joh03]) that

$$(1.36) \quad \mathbb{P}(x_1^1 \leq z_1, \dots, x_1^n \leq z_n) = \det(I - PKP),$$

where $z_i \in \mathcal{X}^i$, K is the correlation kernel given by the Eynard-Mehta Theorem and P is block-diagonal matrix with n diagonal blocks defined so that, for $i = 1, \dots, n$, $P_{i,i}$ has rows and columns indexed by \mathcal{X}^i and is given by $(P_{i,i}v)_j = \mathbf{1}_{x_j^i > z_i} v_j$. This should be compared with an expression like (1.19).

If we go back to thinking about these paths as defining a random point process, then they are given by a measure on kn -point configurations on $\mathcal{X}^1 \cup \dots \cup \mathcal{X}^n$. Therefore, if the process is determinantal, its correlation kernel necessarily has to be a matrix with rows and columns indexed by $\mathcal{X}^1 \cup \dots \cup \mathcal{X}^n$. The Eynard-Mehta Theorem implies moreover that the correlation matrix is partitioned naturally into $n \times n$ blocks, with the (i, j) block having rows indexed by \mathcal{X}^i and columns indexed by \mathcal{X}^j . To see how this structure relates with the extended kernels introduced in Section 1.2.4 for the Airy processes, we make the following observation. An operator T acting on $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$ can be regarded as an operator-valued matrix $(T_{i,j})_{i,j=1,\dots,n}$ with entries $T_{i,j}$ (acting on $L^2(\mathbb{R})$), which acts on $f \in L^2(\mathbb{R})^n$ as $(Tf)_i = \sum_{j=1}^n T_{i,j} f_j$ (more precisely, we are using the fact that $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$ and $\bigoplus_{t \in \{t_1, \dots, t_n\}} L^2(\mathbb{R})$ are isomorphic as Hilbert spaces). Hence an extended kernel formula like (1.19) can be thought of as the determinant of an $n \times n$ matrix whose entries are operators on $L^2(\mathbb{R})$. Similarly, we may think of (1.36) as the determinant of an $n \times n$

matrix whose (i, j) entry maps $L^2(\mathcal{X}^i)$ to $L^2(\mathcal{X}^j)$. Since the Airy processes live on the real line instead of finite sets, this latter spaces are replaced by $L^2(\mathbb{R})$.

In the next section we will explain how these ideas can be used to derive the extended kernel formula for Airy_2 .

1.5.2. Derivation of the Airy_2 process from Dyson Brownian motion. The original derivation of the Airy_2 process was done in [PS02] using quantum statistical mechanical arguments, while Johansson's proof of Theorem 1.1 relies crucially on the connection between LPP and the Robinson-Schensted-Knuth algorithm, which provides a family of discrete non-intersecting paths, the top line of which converges to \mathcal{A}_2 . We will briefly explain the derivation of the Fredholm determinant formula for the Airy_2 process using a different model, the Dyson Brownian motion. We refer the reader to [TW07; TW04; Joh03; Joh06] for more details on the derivation of Airy_2 from non-intersecting paths.

Consider the evolving eigenvalues of an $N \times N$ GUE matrix with each (algebraically independent) entry diffusing according to a stationary Ornstein-Uhlenbeck process. We write the eigenvalues at time t as $\lambda^N(t) = (\lambda_1^N(t), \dots, \lambda_N^N(t))$ so that $\lambda_i(t)$ decreases in i . This eigenvalue diffusion, called the *stationary GUE Dyson Brownian motion*, can be written as the solution of a certain N -dimensional SDE, and it can be shown that it is stationary, with distribution given by the eigenvalue distribution of an $N \times N$ GUE matrix. Moreover, the paths followed by the N eigenvalues almost surely form an ensemble of non-intersecting curves.

Suppose we look at this eigenvalue diffusion at times $t_1 < \dots < t_n$, and we condition the N paths to be pairwise non-intersecting. To investigate the transitions between t_m and t_{m+1} , suppose we condition this eigenvalue diffusion to start at time t_m at $\lambda_i^N(t_m) = x_i$ for some fixed $x_1 < \dots < x_N$, and we also fix destination points $y_1 < \dots < y_N$. Let $p_t(x, y)$ be the transition probability density of a (one-dimensional) Ornstein-Uhlenbeck process from x at time 0 to y at time t . Then, in this setting, the Karlin-McGregor Theorem [KM59] implies that the transition probability density for these N non-intersecting paths to end at the prescribed destination points y_1, \dots, y_N is given by a constant times

$$(1.37) \quad \det[p_{t_{m+1}-t_m}(x_i, y_j)]_{i,j=1}^N.$$

The transition function $p_{t_{m+1}-t_m}$ corresponds then to the matrix W_m in (1.35). Of course, since our paths take values in \mathbb{R} now, we no longer have a matrix, but the Eynard-Mehta Theorem still holds in this setting (see e.g. [TW04]). The functions ϕ_i and ψ_i in (1.35) are related to the (stationary) marginals for λ_i^N , and in this case are equal (due to stationarity) and expressed simply in terms of Hermite polynomials. The result after further computations and using (1.36) is the following [TW04]: given $x_1, \dots, x_n \in \mathbb{R}$,

$$\mathbb{P}(\lambda_1^N(t_1) \leq x_1, \dots, \lambda_1^N(t_n) \leq x_n) = \det(I - f K_{\text{Hrm}, N}^{\text{ext}} f)_{L^2(\{t_1, \dots, t_n\} \times \mathbb{R})},$$

where f is defined as in (1.20) and $K_{\text{Hrm}, N}^{\text{ext}}$ is the *extended Hermite kernel*

$$K_{\text{Hrm}, N}^{\text{ext}}(s, x; t, y) = \begin{cases} \sum_{k=0}^{N-1} e^{k(s-t)} \varphi_k(x) \varphi_k(y) & \text{if } s \geq t, \\ - \sum_{k=N}^{\infty} e^{k(s-t)} \varphi_k(x) \varphi_k(y) & \text{if } s < t, \end{cases}$$

and where $\varphi_k(x) = e^{-x^2/2} p_k(x)$ with p_k the k -th normalized Hermite polynomial (so that $\|\varphi_k\|_2 = 1$). Note that λ_1^N is the top line of our family of non-intersecting paths, so this probability is the same as the probability that all paths stay below the x_i 's. Note also the similarity between this formula and the formula (1.19) for \mathcal{A}_2 . We remark that the scaling of the eigenvalues appearing in the last formula differs by a factor of \sqrt{N} with the one

introduced in Section 1.2.2; the present choice is the one that is naturally associated with the operator D introduced next.

The kernel $K_{\text{Hrm},N}^{\text{ext}}$ has a nice algebraic structure. Writing

$$D = -\frac{1}{2}(\Delta - x^2 + 1),$$

(i.e. $Df(x) = -\frac{1}{2}(f''(x) - (x^2 - 1)f(x))$), the *harmonic oscillator functions* φ_k satisfy $D\varphi_k = k\varphi_k$, and moreover $\{\varphi_k\}_{k \geq 0}$ forms a complete orthonormal basis of $L^2(\mathbb{R})$. Define the *Hermite kernel* as

$$K_{\text{Hrm},N}(x, y) = \sum_{k=0}^{N-1} \varphi_k(x) \varphi_k(y),$$

which is then just the projection onto $\text{span}\{\varphi_0, \dots, \varphi_{N-1}\}$. Then the following formula holds:

$$K_{\text{Hrm},N}^{\text{ext}}(s, x; t, y) = \begin{cases} e^{(s-t)D} K_{\text{Hrm},N}(x, y) & \text{if } s \geq t, \\ -e^{(s-t)D} (I - K_{\text{Hrm},N})(x, y) & \text{if } s < t. \end{cases}$$

Now introduce the rescaled process

$$\tilde{\lambda}_i^N(t) = \sqrt{2}N^{1/6}(\lambda_i^N(N^{-1/3}t) - \sqrt{2N}).$$

Changing variables $x \mapsto \frac{1}{\sqrt{2}N^{1/6}}x + \sqrt{2N}$, $y \mapsto \frac{1}{\sqrt{2}N^{1/6}}y + \sqrt{2N}$ in the kernel accordingly, a calculation gives

$$\mathbb{P}(\tilde{\lambda}_1^N(t_1) \leq x_1, \dots, \tilde{\lambda}_1^N(t_n) \leq x_n) = \det(I - \text{f} \tilde{K}_{\text{Hrm},N}^{\text{ext}} \text{f})_{L^2(\{t_1, \dots, t_n\} \times \mathbb{R})}$$

with

$$\tilde{K}_{\text{Hrm},N}^{\text{ext}}(s, x; t, y) = \begin{cases} e^{(s-t)H_N} \tilde{K}_{\text{Hrm},N}(x, y) & \text{if } s \geq t, \\ -e^{(s-t)H_N} (I - \tilde{K}_{\text{Hrm},N})(x, y) & \text{if } s < t, \end{cases}$$

where $\tilde{K}_{\text{Hrm},N}(x, y) = \frac{1}{\sqrt{2}N^{1/6}} K_{\text{Hrm},N}\left(\frac{x}{\sqrt{2}N^{1/6}} + \sqrt{2N}, \frac{y}{\sqrt{2}N^{1/6}} + \sqrt{2N}\right)$ and the operator $H_N = -\Delta + x + \frac{x^2}{2N^{2/3}}$.

The above rescaling corresponds to focusing in on the top curves of the Dyson Brownian motion. It is known that in the limit $N \rightarrow \infty$, $\tilde{K}_{\text{Hrm},N}$ converges to the Airy kernel K_{Ai} , while it is clear that H_N converges to the *Airy Hamiltonian* H :

$$(1.38) \quad H = -\Delta + x$$

(i.e. $Hf(x) = -f''(x) + xf(x)$). Putting aside precise convergence issues, the result is that

$$(1.39) \quad \lim_{N \rightarrow \infty} \tilde{K}_{\text{Hrm},N}^{\text{ext}}(s, x; t, y) = \begin{cases} e^{(s-t)H} K_{\text{Ai}}(x, y) & \text{if } s \geq t, \\ -e^{(s-t)H} (I - K_{\text{Ai}})(x, y) & \text{if } s < t. \end{cases}$$

The obvious question at this point is what is the relationship between this limit and the extended Airy kernel (1.21). It turns out that they are the same. This is a consequence of the following remark, which implies that the nice structure we saw in $\tilde{K}_{\text{Hrm},N}$ survives in the limit:

Remark 1.8. The shifted Airy functions $\phi_\lambda(x) = \text{Ai}(x - \lambda)$ are the generalized eigenfunctions of the Airy Hamiltonian, as $H\phi_\lambda = \lambda\phi_\lambda$ (we say generalized because $\phi_\lambda \notin L^2(\mathbb{R})$). The Airy kernel K_{Ai} is the projection of H onto its negative generalized eigenspace. This is seen by observing that if we define the operator A to be the *Airy transform*, $Af(x) := \int_{-\infty}^{\infty} dz \text{Ai}(x - z)f(z)$, then $K_{\text{Ai}} = A\bar{P}_0 A^*$, where $\bar{P}_0 f(x) = \mathbf{1}_{x < 0} f(x)$.

In particular, e^{tH} is defined spectrally. Formally, its integral kernel is given by $e^{tH}(x, y) = \int_{-\infty}^{\infty} d\lambda e^{-t\lambda} \text{Ai}(x + \lambda) \text{Ai}(y + \lambda)$. The integral converges when $t < 0$ by the decay properties of the Airy function, but it diverges when $t > 0$ (it can be interpreted as $\delta_{x=y}$ when $t = 0$). Nevertheless, in our formulas e^{tH} will always appear after K_{Ai} when $t > 0$. This has the

effect of restricting the integral to $\lambda > 0$, which converges because the Airy function is bounded.

As a consequence of the above discussion we obtain the following result, see [TW04]:

Theorem 1.9.

$$\tilde{\lambda}_1^N(t) \xrightarrow[N \rightarrow \infty]{} \mathcal{A}_2(t)$$

in the sense of convergence of finite-dimensional distributions.

The extended kernels which define the other Airy processes do not have exactly the same structure. One reason behind this is that, apart from \mathcal{A}_2 and \mathcal{A}_1 , the other processes are not stationary. The other reason is that, as we mentioned, in some cases (for example Airy_1) the processes are obtained after further limiting procedures, which destroy part of the structure. Nevertheless, as we will see later, enough of this structure remains for our purposes.

We have wandered a bit far from the main subject of this survey in the hope that the reader will get a feeling about why extended kernels appear naturally for our processes. In the rest of this article we will deal with formulas which are given as Fredholm determinants of certain operators acting on $L^2(\mathbb{R})$, as opposed $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$. In light of the above discussion, it is slightly surprising that such formulas should exist.

2. FREDHOLM DETERMINANTS

If K is an integral operator acting on $H = L^2(X, d\mu)$ through its kernel

$$(2.1) \quad (Kf)(x) = \int_X K(x, y)f(y)d\mu(y),$$

we define the *Fredholm determinant* by

$$(2.2) \quad \det(I + \lambda K) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_X \cdots \int_X \det [K(x_i, x_j)]_{i,j=1}^n d\mu(x_1) \cdots d\mu(x_n).$$

If $|K(x, y)| \leq B$ for all x, y , and μ is a finite measure, the *Fredholm series* (2.2) converges by Hadamard's inequality,

$$|\det(C_1, \dots, C_n)| \leq \|C_1\| \cdots \|C_n\|$$

where $\|C_i\|$ denotes the Euclidean length of the column vector C_i , since the length of the column vector in $[K(x_i, x_j)]_{i,j=1}^n$ is bounded by $Bn^{1/2}$, and hence the n -th summand in (2.2) is bounded by $\frac{\lambda^n}{n!} B^n n^{n/2}$.

If one is not familiar with the definition (2.2) one might even wonder what it has to do with determinants. Take a matrix $K = [K_{ij}]_{i,j=1}^d$, $d < \infty$, and consider the $d \times d$ determinant $\det(I + \lambda K)$. Clearly it is a polynomial of degree d in λ , $\sum_{n=0}^d a_n \lambda^n$, and its coefficients are given by the rule $a_n = \frac{1}{n!} \partial_\lambda^n \det(I + \lambda K)|_{\lambda=0}$. To compute this, use the rule for differentiating determinants,

$$\partial_\lambda \det(C_1, \dots, C_d) = \sum_{n=1}^d \det(C_1, \dots, \partial_\lambda C_n, \dots, C_d)$$

and the fact that, in our particular case, $C_n(\lambda) = e_n + \lambda K_{\cdot,n}$ is linear in λ and $C_n(0) = e_n$, the n -th unit vector. The result is

$$\begin{aligned} \det(I + \lambda K) = 1 + \lambda \sum_{1 \leq i \leq d} K_{ii} + \lambda^2 \sum_{1 \leq i < j \leq d} \det \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} \\ + \lambda^3 \sum_{1 \leq i < j < k \leq d} \det \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix} + \cdots + \lambda^d \det K. \end{aligned}$$

Replacing the ordered sums with unordered sums gives a factor $1/n!$, and setting $\lambda = 1$ we can see that this is a special case of (2.2). Von Koch's idea [Koc92] was that this formula for the determinant was the natural one to extend to $d = \infty$. Fredholm replaced the integral operator (2.1) on $L^2([0, 1], dx)$ by its discretization $[\frac{1}{n}K(\frac{i}{n}, \frac{j}{n})]_{i,j=1}^n$ to obtain (2.2), which he then used to characterize the solvability of the integral equation $(I + K)u = f$ via the non-vanishing of the determinant of $I + K$.

One can of course imagine other, more intuitive definitions of the determinant. Perhaps

$$(2.3) \quad \det(I + K) = \prod_n (1 + \lambda_n)$$

where λ_n are the eigenvalues of K , counted with multiplicity. Or

$$(2.4) \quad \det(I + \lambda K) = e^{\text{tr} \log(1 + \lambda K)}$$

with the trace

$$(2.5) \quad \text{tr} K = \int d\mu(x) K(x, x).$$

Of course, these definitions require some smallness condition on K , but at least they make apparent the important fact that the determinant is invariant under conjugation $\det(I + M^{-1}KM) = \det(I + K)$, or

$$(2.6) \quad \det(I + K_1 K_2) = \det(I + K_2 K_1),$$

(usually referred to as the cyclic property of determinants) as well as the formula

$$(2.7) \quad \partial_\beta \det(I + K(\beta)) = \det(I + K(\beta)) \text{tr}((I + K(\beta))^{-1} \partial_\beta K(\beta))$$

for $K(\beta)$ depending smoothly on a parameter β .

A more modern way to write (2.2) is

$$(2.8) \quad \det(I + \lambda K) = \sum_{n=0}^{\infty} \lambda^n \text{tr} \Lambda^n(K)$$

where $\Lambda^n(K)$ denotes the action of the tensor product $A \otimes \cdots \otimes A$ on the antisymmetric subspace of $H \otimes \cdots \otimes H$. If P_n denotes the projection onto that subspace and $C_n = P_n \Lambda^n(K) P_n$ then

$$\begin{aligned} C_n(f_1 \otimes \cdots \otimes f_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) A f_{\sigma(1)} \otimes \cdots \otimes A f_{\sigma(n)} \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \int \cdots \int d\mu(y_1) \cdots d\mu(y_n) K(x_1, y_{\sigma(1)}) \cdots K(x_n, y_{\sigma(n)}) f_1(y_1) \cdots f_n(y_n) \end{aligned}$$

which shows that C_n is an integral operator with kernel $\det[K(x_i, x_j)]_{i,j=1}^n$ and hence (2.8) is just a slick way to write (2.2). The advantage of (2.2) is that it can be used directly to define the Fredholm determinant for operators on a general separable Hilbert space, but we will not need this point of view here (see [Sim05] for more details).

The natural notion of smallness for Fredholm determinants turns out to be the trace norm on operators

$$\|K\|_1 := \text{tr} |K|,$$

where $|K| = \sqrt{K^*K}$ is the unique positive square root of the operator K^*K . A (necessarily compact) operator with finite trace norm is called *trace class*. Using the Parseval relation, one can check that for such operators the trace can be defined as

$$\mathrm{tr} K = \sum_{n=1}^{\infty} \langle e_n, K e_n \rangle,$$

as it is basis independent. This works for operators on any separable Hilbert space, and in our setting it can be shown that this definition of trace coincides with (2.5) for K of trace class. The *Hilbert-Schmidt norm* $\|K\|_2 = \sqrt{\mathrm{tr}(|K|^2)}$ is easier to compute,

$$\|K\|_2 = \left(\int dx dy |K(x, y)|^2 \right)^{1/2},$$

and the relation between these norms and the more common operator norm $\|K\|_{\mathrm{op}}$ is

$$\|K\|_{\mathrm{op}} \leq \|K\|_2 \leq \|K\|_1,$$

as well as

$$\|K_1 K_2\|_1 \leq \|K_1\|_2 \|K_2\|_2, \quad \|AK\|_1 \leq \|A\|_{\mathrm{op}} \|K\|_1, \quad \text{and} \quad \|AK\|_2 \leq \|A\|_{\mathrm{op}} \|K\|_2,$$

all of which can be checked easily. Of course, in the latter two A need not be compact. The reason the trace norm is so useful is

Lemma 2.1.

1. (*Lidskii's Theorem*) If K is trace class then $\mathrm{tr} K = \sum_n \lambda_n$, where λ_n are the eigenvalues of K . It follows that the three definitions (2.2), (2.3) and (2.4) are equivalent.
2. $A \mapsto \det(I + A)$ is continuous in trace norm. Explicitly,

$$(2.9) \quad |\det(I + K_1) - \det(I + K_2)| \leq \|K_1 - K_2\|_1 \exp(\|K_1\|_1 + \|K_2\|_1 + 1).$$

Lidskii's theorem is non-trivial and its proofs use heavy function theory, but (2.9) can be explained easily. Let $f(z) = \det(I + \frac{1}{2}(K_1 + K_2) + z(K_1 - K_2))$, so that the left hand side of (2.9) is $|f(\frac{1}{2}) - f(-\frac{1}{2})| \leq \sup_{-1/2 \leq t \leq 1/2} |f'(t)|$. Cauchy's integral formula $f'(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z' - z} dz'$ shows that $\sup_{-1/2 \leq t \leq 1/2} |f'(t)| \leq \frac{1}{R} \sup_{|z| \leq R + \frac{1}{2}} |f(z)|$. The eigenvalues of $\Lambda^n(K)$ are $\lambda_{i_1} \cdots \lambda_{i_n}$, $i_1 < \cdots < i_n$, so $\mathrm{tr} \Lambda^n(K) = \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n}$ and hence $|\mathrm{tr} \Lambda^n(K)| \leq \frac{1}{n!} \|K\|_1^n$, which implies

$$|\det(I + \lambda K)| \leq e^{\lambda \|K\|_1}.$$

Therefore $\sup_{|z| \leq R + \frac{1}{2}} |f(z)| \leq \exp(\frac{1}{2} \|K_1 + K_2\|_1 + (R + \frac{1}{2}) \|K_1 - K_2\|_1)$ and taking $R = \|K_1 - K_2\|_1^{-1}$ gives (2.9).

Examples.

1. (*Gaussian distribution*) A trivial example is $K(x, y) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$. The operator is rank one, so if P_s is the orthogonal projection from $L^2(\mathbb{R}) \rightarrow L^2(s, \infty)$ then by (2.4) we have

$$\det(I - P_s K P_s)_{L^2(\mathbb{R}, dx)} = 1 - \mathrm{tr} P_s K P_s = \int_{-\infty}^s dx \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}.$$

Of course, the Gaussian here could be replaced by an arbitrary density.

2. (*GUE*) Consider the Airy kernel $K_{\mathrm{Ai}}(x, y) = \int_0^\infty dt \mathrm{Ai}(x+t) \mathrm{Ai}(y+t)$ and let $\mathrm{Ai}_t(x) = \mathrm{Ai}(x+t)$ and $H = -\partial_x^2 + x$. Then $H \mathrm{Ai}_t = -t \mathrm{Ai}_t$, the Ai_t , $t \in \mathbb{R}$ are generalized eigenfunctions of H , and K_{Ai} is the orthogonal projection onto the negative eigenspace of

H (see Remark 1.8). Using $\text{Ai}''(x) = x\text{Ai}(x)$, we have $\partial_t \frac{\text{Ai}(x+t)\text{Ai}'(y+t) - \text{Ai}'(x+t)\text{Ai}(y+t)}{y-x} = \text{Ai}(x+t)\text{Ai}(y+t)$, which yields the Christoffel-Darboux formula

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}'(x+t)\text{Ai}(y+t) - \text{Ai}(x+t)\text{Ai}'(y+t)}{y-x}.$$

To show $P_s K_{\text{Ai}} P_s$ is trace class, write $K_{\text{Ai}} = B_0 P_0 B_0$ where

$$(2.10) \quad B_0(x, y) = \text{Ai}(x+y).$$

Then use $\|K_1 K_2\|_1 \leq \|K_1\|_2 \|K_2\|_2$ to get

$$(2.11) \quad \|P_s K_{\text{Ai}} P_s\|_1 \leq \|P_s B_0 P_0\|_2^2 \leq \int_0^\infty \int_s^\infty \text{Ai}^2(x+y) dx dy,$$

which is finite by the following well-known estimates for the Airy function (see (10.4.59-60) in [AS64]):

$$(2.12) \quad |\text{Ai}(x)| \leq C e^{-\frac{2}{3}x^{3/2}} \quad \text{for } x > 0, \quad |\text{Ai}(x)| \leq C \quad \text{for } x \leq 0.$$

The GUE Tracy-Widom distribution is given by

$$F_{\text{GUE}}(s) = \det(I - P_s K_{\text{Ai}} P_s)_{L^2(\mathbb{R}, dx)}.$$

On the face of it, it is not so obvious why such an expression would define a probability distribution function. From (2.11) it is clear that $\lim_{s \rightarrow \infty} \det(I - P_s K_{\text{Ai}} P_s) = 1$. Since $P_s K_{\text{Ai}} P_s$ is a composition of projections, its eigenvalues satisfy $1 \geq \lambda_1(s) \geq \lambda_2(s) \geq \dots \geq 0$. Recall the min-max characterization of eigenvalues

$$\lambda_k(s) = \max_{\dim U=k} \min_{f \in U} \frac{\langle f, P_s K_{\text{Ai}} P_s f \rangle}{\langle f, f \rangle},$$

from which it is apparent that $\lambda_i(s)$ is non-decreasing as s decreases, and hence $\det(I - P_s K_{\text{Ai}} P_s) = \prod_i (1 - \lambda_i(s))$ is non-increasing with decreasing s . In fact, $\lambda_1(s) \nearrow 1$ as $s \searrow -\infty$ since if f is in the negative eigenspace of H , $\langle P_s f, K_{\text{Ai}} P_s f \rangle \rightarrow \langle f, K_{\text{Ai}} f \rangle = \langle f, f \rangle$. This shows that $\det(I - P_s K_{\text{Ai}} P_s) \searrow 0$ as $s \searrow -\infty$ (for an asymptotic expansion of $F_{\text{GUE}}(s)$ as $s \searrow -\infty$ see [BBD08]).

3. (*GOE*) $F_{\text{GOE}}(s) = \det(I - P_s B_0 P_s)_{L^2(\mathbb{R}, dx)}$ where $B_0(x, y)$ is as in (2.10). The key to show that B_0 is trace class in this case is the identity

$$(2.13) \quad \int_{-\infty}^\infty dx \text{Ai}(a+x) \text{Ai}(b-x) = 2^{-1/3} \text{Ai}(2^{-1/3}(a+b))$$

(see, for example, (3.108) in [VS10]). One defines $G_1(x, z) = 2^{1/3} \text{Ai}(2^{1/3}x + z)e^z$ and $G_2(z, y) = e^{-z} \text{Ai}(2^{1/3}y - z)$ and notes that $P_s B_0 P_s = (P_s G_1)(G_2 P_s)$. Then (2.12) allows to show that each of the last two factors has finite Hilbert-Schmidt norm, yielding that $P_s B_0 P_s$ is trace class.

4. (*Airy₁ process*) Recall the Fredholm determinant formula (1.22) for the finite-dimensional distributions of the Airy₁ process. It turns out that the kernel $\text{f}K_1^{\text{ext}}\text{f}$ inside the determinant is not trace class, basically because the heat kernel is not even Hilbert-Schmidt on $L^2([s, \infty))$ for $s \in \mathbb{R}$. Nevertheless, the series (2.2) defining the Fredholm determinant is finite in this case, because one can conjugate the kernel $\text{f}K_1^{\text{ext}}\text{f}$ to something which can be proved to be trace class (see [BFP07]).

The situation in the last example, where the natural expression for a kernel defines an operator which is not trace class, but which is conjugate to a trace class operator, arises often. Here by conjugacy we mean the following: two operators K and \tilde{K} are conjugate if there exists some invertible linear mapping U acting on measurable functions on X such that $K = U \tilde{K} U^{-1}$. Observe that such a pair of operators have the same Fredholm series

expansion (2.2), i.e. $\det(I + K) = \det(I + \tilde{K})$. This allows to extend the manipulations on Fredholm determinants to operators which are conjugate to trace class operators, provided that one is careful in keeping track of the needed conjugations.

The reason we start with the Fredholm expansion (2.2) is that this is the way the determinant usually arises from combinatorial expressions. Sometimes the kernels are not trace class, but this should not bother us so much as long as some version of the formal expression can be shown to converge, for instance as in Example 4 above. Often, it is genuinely difficult to show that the resulting expressions define a probability distribution, and we only know it because they arose this way.

3. BOUNDARY VALUE KERNELS AND CONTINUUM STATISTICS OF AIRY PROCESSES

3.1. Boundary value kernel formulas for finite-dimensional distributions. Recall the formula (1.19) for the finite-dimensional distributions of the Airy_2 process. It is given in terms of the Fredholm determinant of what we call an *extended kernel*, that is, (the kernel of) an operator acting on the “extended space” $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$. Although such formulas have been very useful in the study of models in the KPZ class, they suffer from two problems. First, if one wants to take the number n of times t_i to infinity, a big difficulty appears in the fact that these formulas involve Fredholm determinants on the Hilbert space $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$, and thus the space itself is changing as n grows. Second, these formulas are useful for computing long range properties of the processes (for instance an asymptotic expansion of the covariance of $\mathcal{A}_2(s)$ and $\mathcal{A}_2(t)$ as $|t - s| \rightarrow \infty$, see [Wid04]), but are not suitable for studying short range properties such as regularity of the sample paths.

The second type of Fredholm determinant formula, which is the one we will use for most of the rest of this article, was actually introduced as the original definition of the Airy_2 process by Prähofer and Spohn [PS02]. It is given as follows: for $t_1 < \dots < t_n$ and $x_1, \dots, x_n \in \mathbb{R}$,

$$(3.1) \quad \mathbb{P}(\mathcal{A}_2(t_1) \leq x_1, \dots, \mathcal{A}_2(t_n) \leq x_n) \\ = \det \left(I - K_{\text{Ai}} + \bar{P}_{x_1} e^{(t_1 - t_2)H} \bar{P}_{x_2} e^{(t_2 - t_3)H} \dots \bar{P}_{x_n} e^{(t_n - t_1)H} K_{\text{Ai}} \right)_{L^2(\mathbb{R})},$$

where K_{Ai} is Airy kernel (1.9), H is the Airy Hamiltonian (1.38) and \bar{P}_a denotes the projection onto the interval $(-\infty, a]$:

$$\bar{P}_a f(x) = \mathbf{1}_{x \leq a} f(x).$$

Note that the Fredholm determinant is now computed on the Hilbert space $L^2(\mathbb{R})$ instead of $L^2(\{t_1, \dots, t_n\} \times \mathbb{R})$, which makes taking $n \rightarrow \infty$ at least feasible. Note also that the time increments $t_i - t_{i+1}$ appear explicitly in the formula, which explains why this formula will be more suitable for the study of short range properties. Another advantage of this formula is that it makes apparent that \mathcal{A}_2 is a stationary process.

The equivalence of (1.19) and (3.1) was derived formally in [PS02] and [PS11]. The proof in [PS11] is based in the following idea. As we explained in Section 1.5.1, the extended kernel formula (1.19) can be thought of as the determinant of an $n \times n$ matrix whose entries are operators acting on $L^2(\mathbb{R})$. By rewriting this operator as a sum of an upper-triangular part and lower-triangular part and using algebraic properties of the determinant and the algebraic relationships between the different entries of this matrix, [PS02] showed that the determinant equals the determinant of an operator-valued matrix $I + G$ such that only the first column of G is non-zero. Therefore $\det(I + G)_{L^2(\{t_1, \dots, t_n\} \times \mathbb{R})} = \det(I + G_{1,1})_{L^2(\mathbb{R})}$ (to see this simply pretend the operators in the determinants are matrices), and an explicit calculation of $G_{1,1}$ yields (3.1).

The argument given in [PS11] which we just sketched is almost a complete proof. There are nevertheless some subtleties. For example, it is not a priori obvious that for $s, t > 0$,

e^{-sH} can be applied to the image of $\bar{P}_a e^{-tH}$. Moreover, in order to manipulate Fredholm determinants one needs to check that certain analytical conditions are satisfied (see Section 2). The technical details are discussed in [QR12a], which in fact shows that a formula analogous to (3.1) holds for the Airy₁ process as well. It is given as follows: for $t_1 < \dots < t_n$ and $x_1, \dots, x_n \in \mathbb{R}$,

$$(3.2) \quad \mathbb{P}(\mathcal{A}_1(t_1) \leq x_1, \dots, \mathcal{A}_1(t_n) \leq x_n) \\ = \det \left(I - B_0 + \bar{P}_{x_1} e^{(t_2-t_1)\Delta} \bar{P}_{x_2} e^{(t_3-t_2)\Delta} \dots \bar{P}_{x_n} e^{(t_1-t_n)\Delta} B_0 \right)_{L^2(\mathbb{R})},$$

where B_0 is given by the kernel $B_0(x, y) = \text{Ai}(x+y)$ defined in (1.13) and Δ is the Laplacian operator. Observe that in all but the last factor of the form $e^{s\Delta}$ in the above formula it holds that $s > 0$, in which case $e^{s\Delta}$ is the usual heat kernel. This kernel is ill-defined for $s < 0$, but it turns out that in this case the operator $e^{s\Delta} B_0$ makes sense if defined via the integral kernel

$$(3.3) \quad e^{s\Delta} B_0(x, y) = e^{2s^3/3+s(x+y)} \text{Ai}(x+y+s^2).$$

What we mean by this is that if $s, t > 0$ then, with this definition the semigroup property $e^{t\Delta} e^{-s\Delta} B_0 = e^{(t-s)\Delta} B_0$ holds.

As we will see in Section 3.2, it is fruitful to think of the operator appearing in (3.1) as the solution of certain boundary value problem, so we will refer to formulas like this as *boundary value kernel* formulas. By using (3.3) one can rewrite the definition (1.23) of the extended kernel for \mathcal{A}_1 as

$$K_1^{\text{ext}}(s, x; t, y) = \begin{cases} e^{(t-s)\Delta} B_0(x, y) & \text{if } s \geq t, \\ -e^{(t-s)\Delta} (I - B_0)(x, y) & \text{if } s < t. \end{cases}$$

It becomes clear then that both the extended kernel formula and the boundary value kernel formula for Airy₁ are obtained from the corresponding formulas for Airy₂ by substituting H with $-\Delta$ and K_{Ai} with B_0 . It turns out, as shown in [BCR], that the necessary structure behind these formulas hold for a much wider class of processes, including for instance the stationary GUE Dyson Brownian motion and non-stationary processes like the Airy_{2→1} process, and the Pearcey process [TW06]. For example, for Airy_{2→1} one has [BCR]

$$(3.4) \quad \mathbb{P}(\mathcal{A}_{2 \rightarrow 1}(t_1) \leq x_1, \dots, \mathcal{A}_{2 \rightarrow 1}(t_n) \leq x_n) \\ = \det \left(I - K_{2 \rightarrow 1}^{t_1} + \bar{P}_{\tilde{x}_1} e^{(t_2-t_1)\Delta} \bar{P}_{\tilde{x}_2} \dots e^{(t_n-t_{n-1})\Delta} \bar{P}_{\tilde{x}_n} e^{(t_1-t_n)\Delta} K_{2 \rightarrow 1}^{t_1} \right)_{L^2(\mathbb{R})},$$

where $\tilde{x}_i = x_i - t_i^2 \mathbf{1}_{t_i \leq 0}$ and $K_{2 \rightarrow 1}^t(x, y) = K_{2 \rightarrow 1}^{\text{ext}}(t, x; t, y)$ with $K_{2 \rightarrow 1}^{\text{ext}}$ as in (1.24).

Interestingly, it is shown in [BCR] that in a setting corresponding to discrete non-intersecting paths, analogous boundary value kernel formulas can be obtained directly from applying the Karlin-McGregor formula (1.37) (or rather its combinatorial analogue, the Lindström-Gessel-Viennot Theorem [Lin73; GV85]), bypassing the direct application of the Eynard-Mehta Theorem. In the case of the Airy₂ process, a suitable limit of a discrete family of non-intersecting should lead to (3.1) (cf. (1.39)). Such a procedure does not seem to work for the Airy₁ process. In fact, in that case the determinantal process used to derive (1.23) is signed (in the sense that the measure defined by the analog of (1.35) is signed), see [BFPS07], and hence it is not clear how to associate directly to it a family of non-intersecting paths.

As we will see below, the integral kernels of the operators appearing inside the Fredholm determinants in (3.1), (3.2) and (3.4) can be expressed simply in terms of hitting probabilities of Brownian motion. In other words, hitting probabilities of curves by \mathcal{A}_2 , \mathcal{A}_1 and $\mathcal{A}_{2 \rightarrow 1}$ can be expressed in terms of Fredholm determinants of the analogous hitting probabilities for Brownian motion. Given the above discussion (and the discussion in Section 1.5.2), this is not entirely surprising in the case of \mathcal{A}_2 , as it follows from the non-intersecting nature

of systems of Brownian paths that can be used to approximate \mathcal{A}_2 . For the same reason, it is surprising in a sense that the same structure is present in \mathcal{A}_1 .

3.2. Continuum statistics and boundary value problems. Consider the following problem: compute the probability that, inside a finite interval $[\ell, r]$, the Airy_2 process lies below a given function g . The obvious way to proceed is to take a fine mesh $\ell = t_1 < t_2 < \dots < t_n = r$ of the interval $[\ell, r]$, take $x_i = g(t_i)$, and attempt to take a limit as $n \rightarrow \infty$ in the formula for the finite dimensional distributions of \mathcal{A}_2 ,

$$(3.5) \quad \mathbb{P}(\mathcal{A}_2(t_1) \leq g(t_1), \dots, \mathcal{A}_2(t_n) \leq g(t_n)) \\ = \det \left(I - K_{\text{Ai}} + \bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \dots \bar{P}_{g(t_n)} e^{(t_n-t_1)H} K_{\text{Ai}} \right).$$

Here the Fredholm determinant is computed on the Hilbert space $L^2(\mathbb{R})$, which we will omit from the subscript in the sequel. By Theorem 1.1, \mathcal{A}_2 has a continuous version, and hence $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{A}_2(t_1) \leq g(t_1), \dots, \mathcal{A}_2(t_n) \leq g(t_n)) = \mathbb{P}(\mathcal{A}_2(t) \leq g(t) \text{ for } t \in [\ell, r])$. To study the right hand side of (3.5) we need to compute the limit of the operator appearing inside the determinant. Observe that the last exponential equals $e^{(r-\ell)H}$, and hence does not depend on n . On the other hand, for $s < t$ the operator $e^{(s-t)H}$ can be thought of as mapping a function f to the solution $u(t, \cdot)$ at time t of the PDE $\partial_t u + Hu = 0$ with initial condition $u(s, \cdot) = f(\cdot)$. Therefore the operator

$$(3.6) \quad \bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \dots e^{(t_{n-1}-t_n)H} \bar{P}_{g(t_n)}$$

can be thought of as solving the same PDE (backwards in time) on the interval $[\ell, r]$ with the additional condition that all the mass above $g(t_i)$ is removed at each of the discrete times t_i . Note that the PDE is solved backwards because, if we apply this operator to a function on its right, we first apply $\bar{P}_{g(t_n)}$, then $e^{(t_{n-1}-t_n)H}$, then $\bar{P}_{g(t_{n-1})}$, and so on. Since solving the PDE $\partial_t u + Hu = 0$ forward or backwards in time gives the same answer, if we want to think of (3.6) as being solved forward in time, all we need to do is reverse the order in which the $g(t_i)$ appear. The result is the following. Given $g \in H^1([\ell, r])$ (i.e. both g and its derivative are in $L^2([\ell, r])$), define an operator $\Theta_{[\ell, r]}^g$ acting on $L^2(\mathbb{R})$ as follows: $\Theta_{[\ell, r]}^g f(\cdot) = u(r, \cdot)$, where $u(r, \cdot)$ is the solution at time r of the boundary value problem

$$\begin{aligned} \partial_t u + Hu &= 0 \quad \text{for } x < g(t), \quad t \in (\ell, r) \\ u(\ell, x) &= f(x) \mathbf{1}_{x < g(\ell)} \\ u(t, x) &= 0 \quad \text{for } x \geq g(t). \end{aligned}$$

Further, define $\hat{g}(t) = g(\ell + r - t)$. Then

$$(3.7) \quad \left\| \bar{P}_{g(t_1)} e^{(t_1-t_2)H} \bar{P}_{g(t_2)} e^{(t_2-t_3)H} \dots \bar{P}_{g(t_n)} e^{(t_n-t_1)H} K_{\text{Ai}} - \Theta_{[\ell, r]}^{\hat{g}} e^{(t_n-t_1)H} K_{\text{Ai}} \right\|_1 \xrightarrow{n \rightarrow \infty} 0.$$

Since the convergence holds in trace class norm, (3.7) can be used to answer the question with which we started this subsection:

Theorem 3.1 ([CQR13], Theorem 2).

$$(3.8) \quad \mathbb{P}(\mathcal{A}_2(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det \left(I - K_{\text{Ai}} + \Theta_{[\ell, r]}^g e^{(r-\ell)H} K_{\text{Ai}} \right).$$

Observe that we have written g instead of \hat{g} in (3.8). We may do this because the Airy_2 is invariant under time reversal, so we can replace g by \hat{g} on the left hand side.

The limit (3.7) is proved in Proposition 3.2 of [CQR13] (in fact only along the dyadic sequence $n_k = 2^k$, but this is enough for deducing Theorem 3.1). The proof is based on the following probabilistic representation of the solutions of the above boundary value problem:

if $\Theta_{[\ell,r]}^g(x, y)$ denotes the integral kernel of $\Theta_{[\ell,r]}^g$, then

$$(3.9) \quad \Theta_{[\ell,r]}^g(x, y) = e^{\ell x - r y + (r^3 - \ell^3)/3} \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}} \cdot \mathbb{P}_{\hat{b}(\ell)=x-\ell^2, \hat{b}(r)=y-r^2} \left(\hat{b}(s) \leq g(s) - s^2 \text{ on } [\ell, r] \right),$$

where the probability is computed with respect to a Brownian bridge $\hat{b}(s)$ from $x - \ell^2$ at time ℓ to $y - r^2$ at time r and with diffusion coefficient 2. This formula is Theorem 3 of [CQR13], its proof is based on an application of the Feynman-Kac and Cameron-Martin-Girsanov formulas.

The argument that proves Theorem 3.1 can be adapted to obtain a similar result for Airy_1 . Fix $\ell < r$. Given $g \in H^1([\ell, r])$, define an operator $\Lambda_{[\ell,r]}^g$ acting on $L^2(\mathbb{R})$ as follows: $\Lambda_{[\ell,r]}^g f(\cdot) = u(r, \cdot)$, where $u(r, \cdot)$ is the solution at time r of the boundary value problem

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{for } x < g(t), \quad t \in (\ell, r) \\ u(\ell, x) &= f(x) \mathbf{1}_{x < g(\ell)} \\ u(t, x) &= 0 \quad \text{for } x \geq g(t). \end{aligned}$$

Theorem 3.2 ([QR12a], Theorem 4).

$$(3.10) \quad \mathbb{P}(\mathcal{A}_1(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det \left(I - B_0 + \Lambda_{[\ell,r]}^g e^{-(r-\ell)\Delta} B_0 \right).$$

Although the proof of this result is similar to the proof for the Airy_2 case, the argument is a bit more involved because, as written, the operator in the determinant is not trace class, so one needs to conjugate appropriately. Of course, similar arguments should allow one to obtain continuum statistics formulas for other processes for which boundary value kernel formulas are available (see [BCR] for the case of stationary GUE Dyson Brownian motion).

The operator $\Lambda_{[\ell,r]}^g$ also has a simple representation in terms of Brownian motion (see [QR12a]), which has recently been used in [FF13] to verify numerically the experimental values obtained in [TS12] for the persistence probabilities of Airy_1 . The negative persistence exponent is defined by

$$\mathbb{P}(\mathcal{A}_1(t) \leq m, \quad 0 \leq t \leq L) \sim e^{-\kappa_- L}$$

where m is the mean of F_{GOE} . Takeuchi has measured $\kappa_- \approx 3.2 \pm 0.2$ in computer simulations of the Eden model [TS12]. Ferrari and Frings [FF13] have computed numerically (3.10) finding

$$\kappa \approx 2.9,$$

which is fairly close. Note that Takeuchi has also measured the positive persistence probabilities $\mathbb{P}(\mathcal{A}_1(t) \geq m, \quad 0 \leq t \leq L) \sim e^{-\kappa_+ L}$. An interesting question is whether there exists a simple enough mathematical formula to check such a thing.

4. APPLICATIONS

In this section we will describe some applications of the boundary value kernel formulas for Airy processes which were introduced in the previous section. The first two applications refer to asymptotic statistics for directed polymers and LPP, while the next two involve respectively the Airy_1 and $\text{Airy}_{2 \rightarrow 1}$ processes.

4.1. Point-to-line LPP and GOE. Recall the variational formula (1.25) relating the Airy₂ process with the Tracy-Widom GOE distribution:

$$(4.1) \quad \mathbb{P}\left(\sup_{x \in \mathbb{R}} \{\mathcal{A}_2(x) - x^2\} \leq m\right) = F_{\text{GOE}}(4^{1/3}m).$$

As we explained in Section 1.2.3, Johansson's proof [Joh03] was very indirect, relying on the convergence of the spatial fluctuations of point-to-point LPP to \mathcal{A}_2 together with (1.15) and (1.17).

A direct proof of this variational formula was provided in [CQR13], based on the continuum statistics formula given in Theorem 3.1. An interesting consequence of this derivation was that it allowed to identify the factor of $4^{1/3}$ on the right hand side of the identity, which had been lost in Johansson's argument in the process of translating between the available results at the time (see Section 2 of [CQR13] for an account of how to get the correct factor directly from LPP).

We will explain next the derivation of the formula, skipping some details. We rewrite the desired probability as

$$\lim_{L \rightarrow \infty} \mathbb{P}(\mathcal{A}_2(t) \leq m + t^2 \ \forall t \in [-L, L]).$$

For fixed $L > 0$, Theorem 3.1 implies that this probability is given by

$$(4.2) \quad \det(I - K_{\text{Ai}} + \Theta_L e^{2LH} K_{\text{Ai}}),$$

where

$$\Theta_L = \Theta_{[-L, L]}^{g(t)=t^2+m}.$$

The nice thing is that the choice of $g(t) = t^2 + m$ is the simplest possible from the point of view of explicit calculations, because it cancels exactly the parabola appearing on the right hand side of (3.9). The probability appearing in that formula is then reduced to the probability of a Brownian bridge staying below level m , and this is easy to compute using the reflection principle (method of images):

$$(4.3) \quad \begin{aligned} & \mathbb{P}_{\hat{b}(-L)=x-L^2, \hat{b}(L)=y-L^2}(\hat{b}(s) \leq m \text{ on } [-L, L]) \\ &= 1 - \mathbb{P}_{\hat{b}(-L)=x-L^2, \hat{b}(L)=y-L^2}(\hat{b}(s) > m \text{ for some } s \in [-L, L]) \\ &= 1 - e^{-(x-m-L^2)(y-m-L^2)/2L} \end{aligned}$$

(we leave the simple computation to the reader, alternatively see page 67 of [BS02]). Putting this back in Θ_L gives

$$(4.4) \quad \Theta_L = \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2} - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2},$$

where R_L is the reflection term

$$(4.5) \quad R_L(x, y) = \frac{1}{\sqrt{8\pi L}} e^{-(x+y-2m-2L^2)^2/8L - (x+y)L+2L^3/3}.$$

The e^{-2LH} in the first term in Θ_L comes from the 1 in (4.3) and appears by either reversing the use of the Cameron-Martin-Girsanov and Feynman-Kac formulas in the derivation of (4.4) or by an explicit computation of the integral kernel of $e^{-(r-\ell)H}$ as

$$e^{-(r-\ell)H}(x, y) = e^{\ell x - r y + (r^3 - \ell^3)/3} \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}}.$$

Referring to (4.2), we have by the cyclic property of determinants (2.6) and the identity $e^{2LH} K_{\text{Ai}} = (e^{LH} K_{\text{Ai}})^2$ (which follows from Remark 1.8) that

$$(4.6) \quad \mathbb{P}(\mathcal{A}_2(t) \leq t^2 + m \text{ for } t \in [-L, L]) = \det(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_L e^{LH} K_{\text{Ai}}).$$

To obtain the $L \rightarrow \infty$ asymptotics, we decompose Θ_L so as to expose the two limiting terms, as well as a remainder term Ω_L :

$$(4.7) \quad \Theta_L = e^{-2LH} - R_L + \Omega_L,$$

where $\Omega_L = (R_L - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}) - (e^{-2LH} - \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2})$. It is shown in [CQR13] that

$$(4.8) \quad \|e^{LH} K_{\text{Ai}} \Omega_L e^{LH} K_{\text{Ai}}\|_1 \xrightarrow{L \rightarrow \infty} 0.$$

The proof amounts essentially to asymptotic analysis involving the Airy function. In view of this fact and the decomposition (4.7), and since $e^{LH} K_{\text{Ai}} e^{-2LH} e^{LH} K_{\text{Ai}} = K_{\text{Ai}}$, we see that the key point is the limiting behaviour in L of

$$e^{LH} K_{\text{Ai}} R_L e^{LH} K_{\text{Ai}}.$$

To explain how this last product can be computed we will proceed in a slightly formal manner through an argument based on the Baker-Campbell-Hausdorff formula, as done for a related problem in [QR13] (see Section 4.4). Since K_{Ai} is a projection and H leaves K_{Ai} invariant, we will pretend that e^{LH} and K_{Ai} commute, so we have to compute the limit of $e^{LH} R_L e^{LH}$. Define the *reflection operator* ϱ_m by

$$\varrho_m f(x) = f(2m - x).$$

Then the operator R_L defined in (4.5) can be rewritten as

$$(4.9) \quad R_L = e^{(2L^3)/3} e^{-L\xi} \varrho_{m+L^2} e^{2L\Delta} e^{-L\xi} = e^{(2L^3)/3} e^{-L\xi} e^{L\Delta} \varrho_{m+L^2} e^{L\Delta} e^{-L\xi}.$$

Here $e^{r\xi}$ (ξ stands for a generic variable) denotes the multiplication operator $(e^{r\xi} f)(x) = e^{rx} f(x)$. The second equality follows from the reflection principle applied to the heat kernel.

The following identities will be useful, where $[\cdot, \cdot]$ denotes commutator:

$$[H, \Delta] = [\xi, \Delta] = -2\nabla, \quad [H, \nabla] = [\xi, \nabla] = -I, \quad [H, \xi] = -2\nabla.$$

If A and B are two operators such that $[A, [A, B]] = c_1 I$ and $[B, [A, B]] = c_2 I$ for some $c_1, c_2 \in \mathbb{R}$, then the Baker-Campbell-Hausdorff formula reads

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]}.$$

Using this we have

$$e^{-L\xi} e^{L\Delta} = e^{L^3/6} e^{L\Delta+L^2\nabla-L\xi}.$$

Using the Baker-Campbell-Hausdorff formula again we deduce that

$$e^{LH} e^{-L\xi} e^{L\Delta} = e^{L^3/6} e^{LH} e^{L\Delta+L^2\nabla-L\xi} = e^{-L^3/3} e^{L^2\nabla},$$

while an analogous computation yields

$$e^{L\Delta} e^{-L\xi} e^{LH} = e^{-L^3/3} e^{-L^2\nabla}.$$

Employing these identities on the right hand side of (4.9) yields

$$e^{LH} R_L e^{LH} = e^{L^2\nabla} \varrho_{m+L^2} e^{-L^2\nabla}.$$

Since $e^{r\nabla}$ is the shift operator $(e^{r\nabla} f)(x) = f(x+r)$, we have $e^{r\nabla} \varrho_m = \varrho_m e^{-r\nabla} = \varrho_{m-r/2}$, and we obtain

$$e^{LH} R_L e^{LH} = \varrho_m.$$

Remarkably, the result does not depend on L . The conclusion from using this, (4.8) and (4.6) in (4.7) and taking $L \rightarrow \infty$ is that

$$(4.10) \quad \mathbb{P}(\mathcal{A}_2(t) \leq t^2 + m \text{ for all } t \in \mathbb{R}) = \det(I - K_{\text{Ai}} \varrho_m K_{\text{Ai}}).$$

The use of the Baker-Campbell-Formula in the derivation of this identity can be replaced by an explicit integral calculation (see the proof of Proposition 1.3 of [CQR13]).

To finish our proof of (4.1) we need to show that the right hand side of (4.10) equals $F_{\text{GOE}}(4^{1/3}m)$. Recall the definition of the kernel $B_0(x, y) = \text{Ai}(x + y)$ and observe that $K_{\text{Ai}} = B_0 P_0 B_0$. Recall also that the shifted Airy functions form a generalized orthonormal basis of $L^2(\mathbb{R})$ (see Remark 1.8), which implies that $B_0^2 = I$. Therefore we can use the cyclic property of determinants (2.6) to deduce that

$$\det(I - K_{\text{Ai}} \varrho_m K_{\text{Ai}}) = \det(I - P_0 B_0 \varrho_m B_0 P_0).$$

Now

$$B_0 \varrho_m B_0(x, y) = \int_{-\infty}^{\infty} d\lambda \text{Ai}(x + \lambda) \text{Ai}(2m - \lambda + y),$$

and using the identity (2.13) we deduce that

$$(4.11) \quad B_0 \varrho_m B_0(x, y) = \tilde{B}_m(x, y) := 2^{-1/3} \text{Ai}(2^{-1/3}(x + y + 2m)),$$

and thus

$$\det(I - K_{\text{Ai}} \varrho_m K_{\text{Ai}}) = \det(I - P_0 \tilde{B}_m P_0).$$

Performing the change of variables $x \mapsto 2^{1/3}x$, $y \mapsto 2^{1/3}y$ in the series defining the last Fredholm determinant shows that the determinant on the right hand side of (4.10) equals $\det(I - P_0 B_{4^{1/3}m} P_0)$, which is $F_{\text{GOE}}(4^{1/3}m)$ by (1.14).

4.2. Endpoint distribution of directed polymers. In the setting of geometric LPP (see Section 1.2.3), consider the random variable

$$\kappa_N = \min \left\{ k \in \{-N, \dots, N\} : \sup_{j=-N, \dots, k} L_N^{\text{point}}(j) = \sup_{j=-N, \dots, N} L_N^{\text{point}}(j) \right\}.$$

κ_N corresponds to the location of the endpoint of the maximizing path in point-to-line LPP.

Interest in the scaling properties and distribution of this random variable goes back at least to the early 1990's. One can also consider the analogous random variable in the setting of directed random polymers, but due to the KPZ universality conjecture one expects that the asymptotic behavior and statistics are the same as in LPP. Mézard and Parisi [MP92] considered the polymer case and derived non-rigorously the scaling relation

$$|\kappa_N| \sim N^{2/3}$$

(c.f. (1.6)). In view of this we define the rescaled endpoint

$$\mathcal{T}_N = c_3^{-1} N^{-2/3} \kappa_N,$$

where c_3 is the constant appearing in (1.18). Recalling the definition of the rescaled point-to-point last passage time (1.18) as the linear interpolation of the values given by

$$H_N^{\text{point}}(t) = \frac{1}{c_2 N^{1/3}} \left[L^{\text{point}}(N + c_3^{-1} N^{-2/3} t, N - c_3^{-1} N^{-2/3} t) - c_1 N \right]$$

for t such that $c_3^{-1} N^{-2/3} t \in \{-N, \dots, N\}$ we deduce that

$$\mathcal{T}_N = \min \left\{ t \in \mathbb{R} : \sup_{s \leq t} H_N^{\text{point}}(s) = \sup_{s \in \mathbb{R}} H_N^{\text{point}}(s) \right\}.$$

Recalling that $H_N(t)$ converges to $\mathcal{A}_2(t) - t^2$ by Theorem 1.1 it becomes clear that \mathcal{T}_N should converge to the point where $\mathcal{A}_2(t) - t^2$ attains its maximum. In fact, this is what Johansson proved, although he had to make a (very reasonable) technical assumption on the Airy_2 process which he was not able to prove with the tools available at the time:

Theorem 4.1 ([Joh03]). *Assume that the process $\mathcal{A}_2(t) - t^2$ attains its maximum at a unique point and let*

$$\mathcal{T} = \arg \max_{t \in \mathbb{R}} \{\mathcal{A}_2(t) - t^2\}.$$

Then

$$\mathcal{T}_N \xrightarrow[N \rightarrow \infty]{} \mathcal{T}$$

in the sense of convergence in distribution.

Although the result is of course very interesting, as it shows that the limiting endpoint distribution exists (under the technical assumption), it gives no information on the distribution of \mathcal{T} . Quoting Johansson [Joh06], for all we know \mathcal{T} could be Gaussian. Nevertheless, from KPZ universality one expects that this is not the case. For example, Halpin-Healy and Zhang [HHZ95] conjectured on the basis of analogy with the argmax of a Brownian motion minus a parabola (for which one has a complete analytical solution, see [Gro89]), that the tails of \mathcal{T} decay like e^{-ct^3} , which of course rules out Gaussian behavior.

It turns out that the distribution of \mathcal{T} can be computed explicitly through an argument based on the continuum statistics formula of Theorem 3.1. This was done in [MFQR13], where in fact the joint density of

$$\mathcal{T} = \arg \max_{t \in \mathbb{R}} \{\mathcal{A}_2(t) - t^2\} \quad \text{and} \quad \mathcal{M} = \max_{t \in \mathbb{R}} \{\mathcal{A}_2(t) - t^2\}$$

was computed. Moreover, the argument implies that the maximum of $\mathcal{A}_2(t) - t^2$ is attained at a unique point, thus completing the proof of Theorem 4.1. The uniqueness of the maximum was also proved slightly earlier by Corwin and Hammond [CH12] using completely different techniques, and a proof for general stationary processes is now available [Pim12].

The computation is as follows. For simplicity we will assume the uniqueness of the maximizing point of $\mathcal{A}_2(t) - t^2$, and will explain later how the uniqueness can actually be obtained from this argument. Let $(\mathcal{M}_L, \mathcal{T}_L)$ denote the maximum and the location of the maximum of $\mathcal{A}_2(t) - t^2$ restricted to $t \in [-L, L]$, and let f_L be the joint density of $(\mathcal{M}_L, \mathcal{T}_L)$. By results of [CH11], the joint density $f(m, t)$ of \mathcal{M}, \mathcal{T} is well approximated by $f_L(m, t)$,

$$f(t, m) = \lim_{L \rightarrow \infty} f_L(t, m).$$

By definition,

$$f_L(t, m) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \delta} \mathbb{P}(\mathcal{M}_L \in [m, m + \varepsilon], \mathcal{T}_L \in [t, t + \delta]),$$

provided that the limit exists. The main contribution in the above expression comes from paths entering the space-time box $[t, t + \delta] \times [m, m + \varepsilon]$ and staying below the level m outside the time interval $[t, t + \delta]$. More precisely, if we denote by $\underline{D}_{\varepsilon, \delta}$ and $\overline{D}_{\varepsilon, \delta}$ the sets

$$\begin{aligned} \underline{D}_{\varepsilon, \delta} = \left\{ \mathcal{A}_2(s) - s^2 \leq m, \quad s \in [t, t + \delta]^c, \quad \mathcal{A}_2(s) - s^2 \leq m + \varepsilon, \quad s \in [t, t + \delta], \right. \\ \left. \mathcal{A}_2(s) - s^2 \in [m, m + \varepsilon] \text{ for some } s \in [t, t + \delta] \right\}, \end{aligned}$$

and

$$\overline{D}_{\varepsilon, \delta} = \left\{ \mathcal{A}_2(s) - s^2 \leq m + \varepsilon, \quad s \in [-L, L], \quad \mathcal{A}_2(s) - s^2 \in [m, m + \varepsilon] \text{ for some } s \in [t, t + \delta] \right\},$$

then

$$\underline{D}_{\varepsilon, \delta} \subseteq \{\mathcal{M}_L \in [m, m + \varepsilon], \mathcal{T}_L \in [t, t + \delta]\} \subseteq \overline{D}_{\varepsilon, \delta}.$$

Letting $\underline{f}(t, m) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \delta} \mathbb{P}(\underline{D}_{\varepsilon, \delta})$ and defining $\overline{f}(t, m)$ analogously (with $\overline{D}_{\varepsilon, \delta}$ instead of $\underline{D}_{\varepsilon, \delta}$) we deduce that $\underline{f}(t, m) \leq f(t, m) \leq \overline{f}(t, m)$. In what follows we will

compute $f(t, m)$. It will be clear from the argument that for $\bar{f}(t, m)$ we get the same limit. The conclusion is that

$$f_L(t, m) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \delta} \mathbb{P}(\underline{D}_{\varepsilon, \delta}).$$

We rewrite this last equation as

$$f_L(t, m) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \delta} \left[\mathbb{P}(\mathcal{A}_2(s) \leq h_{\varepsilon, \delta}(s), s \in [-L, L]) - \mathbb{P}(\mathcal{A}_2(s) \leq h_{0, \delta}(s), s \in [-L, L]) \right],$$

where

$$h_{\varepsilon, \delta}(s) = s^2 + m + \varepsilon \mathbf{1}_{s \in [t, t+\delta]}.$$

These two probabilities have explicit Fredholm determinant formulas by Theorem 3.1. We get, using the cyclic property of determinants as in (4.6),

$$f_L(t, m) = \lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon \delta} \left[\det \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{\varepsilon, \delta}} e^{LH} K_{\text{Ai}} \right) - \det \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{0, \delta}} e^{LH} K_{\text{Ai}} \right) \right].$$

The limit in ε becomes a derivative

$$f_L(t, m) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \partial_{\beta} \det \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{\beta, \delta}} e^{LH} K_{\text{Ai}} \right) \Big|_{\beta=0},$$

which in turn gives a trace by (2.7),

$$(4.12) \quad f_L(t, m) = \det \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{0, \delta}} e^{LH} K_{\text{Ai}} \right) \cdot \lim_{\delta \rightarrow 0} \frac{1}{\delta} \text{tr} \left[\left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{0, \delta}} e^{LH} K_{\text{Ai}} \right)^{-1} e^{LH} K_{\text{Ai}} \left[\partial_{\beta} \Theta_{[-L, L]}^{h_{\beta, \delta}} \right]_{\beta=0} e^{LH} K_{\text{Ai}} \right].$$

One has to check here that the required limits hold in trace class norm, see [MFQR13]. Note that $h_{0, \delta} = g_m$, where g_m is the parabolic barrier

$$g_m(s) = s^2 + m,$$

so in particular the determinant and the first factor inside the trace do not depend on δ . We know moreover from the arguments in Section 4.1 that

$$\lim_{L \rightarrow \infty} \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{0, \delta}} e^{LH} K_{\text{Ai}} \right) = I - K_{\text{Ai}} \varrho_m K_{\text{Ai}}$$

in trace norm. In particular, we have

$$\lim_{L \rightarrow \infty} \det \left(I - K_{\text{Ai}} + e^{LH} K_{\text{Ai}} \Theta_{[-L, L]}^{h_{0, \delta}} e^{LH} K_{\text{Ai}} \right) = F_{\text{GOE}}(4^{1/3} m).$$

The next step is to compute $\partial_{\beta} \Theta_{[-L, L]}^{h_{\beta, \delta}} \Big|_{\beta=0}$. Recalling that $h_{0, \delta}(s) = g_m(s) = s^2 + m$ and also $h_{\varepsilon, \delta}(s) = g_{m+\varepsilon}(s)$ for $s \in [t, t+\delta]$ we have, by the semigroup property,

$$\Theta_{[-L, L]}^{h_{\varepsilon, \delta}} - \Theta_{[-L, L]}^{h_{0, \delta}} = \Theta_{[-L, t]}^{g_m} \left[\Theta_{[t, t+\delta]}^{g_{m+\varepsilon}} - \Theta_{[t, t+\delta]}^{g_m} \right] \Theta_{[t+\delta, L]}^{g_m}.$$

Computing the desired derivative involves just the middle bracket, which we note corresponds to the same boundary value problem as in Section 4.1, only at two different levels m and $m+\varepsilon$. Since we have explicit formulas, the derivative can be computed explicitly. The computation is slightly tedious, and the only delicate part is to justify that the necessary limits occur in trace class norm, we refer to [MFQR13] for the details.

Going back to (4.12), we recall that the trace is linear and continuous under the trace class norm topology, so in view of the preceding discussion we have

$$(4.13) \quad \lim_{L \rightarrow \infty} f_L(t, m) = F_{\text{GOE}}(4^{1/3}m) \operatorname{tr} \left[(I - K_{\text{Ai}} \varrho_m K_{\text{Ai}})^{-1} \lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} \frac{1}{\delta} e^{LH} K_{\text{Ai}} \left[\partial_\beta \Theta_{[-L, L]}^{h_{\beta, \delta}} \right]_{\beta=0} e^{LH} K_{\text{Ai}} \right].$$

Once again we need to compute limits, again taking care that they hold in trace class norm as necessary. We skip the details and just write down the result, (2.12) in [MFQR13]:

$$\lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} \frac{1}{\delta} e^{LH} K_{\text{Ai}} \left[\partial_\beta \Theta_{[-L, L]}^{h_{\beta, \delta}} \right]_{\beta=0} e^{LH} K_{\text{Ai}} = \Psi,$$

where

$$\Psi(x, y) = B_0 P_0 \psi_{t, m}(x) B_0 P_0 \psi_{-t, m}(y)$$

and

$$\psi_{t, m}(x) = 2e^{t^3 + (m+x)t} \left[\operatorname{Ai}'(m + t^2 + x) + t \operatorname{Ai}(m + t^2 + x) \right]$$

(we remark that we have written these formulas in a slightly different way compared to [MFQR13], but the reader should have no problem translating between the formulas). The limit in δ is relatively straightforward, while the limit in L involves an argument similar to the one used in Section 4.1. Using this formula in (4.13) the trace becomes

$$\operatorname{tr} \left[(I - K_{\text{Ai}} \varrho_m K_{\text{Ai}})^{-1} \Psi \right] = \langle (I - K_{\text{Ai}} \varrho_m K_{\text{Ai}})^{-1} B_0 P_0 \psi_{t, m}, B_0 P_0 \psi_{-t, m} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes inner product in $L^2(\mathbb{R})$.

It only remains to simplify the expression. We first use (4.11) and the facts that $K_{\text{Ai}} = B_0 P_0 B_0$, $B_0^2 = I$ and $B_0^* = B_0$ to write

$$\begin{aligned} \langle (I - K_{\text{Ai}} \varrho_m K_{\text{Ai}})^{-1} B_0 P_0 \psi_{t, m}, B_0 P_0 \psi_{-t, m} \rangle &= \langle (I - B_0 P_0 B_m P_0 B_0)^{-1} B_0 P_0 \psi_{t, m}, B_0 P_0 \psi_{-t, m} \rangle \\ &= \langle B_0 (I - P_0 B_m P_0)^{-1} P_0 \psi_{t, m}, B_0 P_0 \psi_{-t, m} \rangle \\ &= \langle (I - P_0 B_m P_0)^{-1} P_0 \psi_{t, m}, P_0 \psi_{-t, m} \rangle. \end{aligned}$$

Next we introduce the scaling operator $Sf(x) = f(2^{1/3}x)$. One can check easily that $S^{-1} = 2^{1/3}S^*$ and that P_0 commutes with S and S^{-1} . We also have $SB_m S^{-1} = B_{4^{1/3}m}$. Thus writing $\tilde{m} = 2^{-1/3}m$ we get

$$\begin{aligned} \langle (I - P_0 B_m P_0)^{-1} P_0 \psi_{t, m}, P_0 \psi_{-t, m} \rangle &= \langle (I - S^{-1} P_0 B_{2\tilde{m}} P_0 S)^{-1} P_0 \psi_{t, m}, P_0 \psi_{-t, m} \rangle \\ &= \langle S^{-1} (I - P_0 B_{2\tilde{m}} P_0)^{-1} P_0 S \psi_{t, m}, P_0 \psi_{-t, m} \rangle \\ &= 2^{1/3} \langle (I - P_0 B_{2\tilde{m}} P_0)^{-1} P_0 S \psi_{t, m}, P_0 S \psi_{-t, m} \rangle. \end{aligned}$$

which is equal to $2^{1/3} \gamma(t, 4^{1/3}m)$.

Using this formula in (4.13) yields the joint density of \mathcal{T} and \mathcal{M} . Define the resolvent kernel

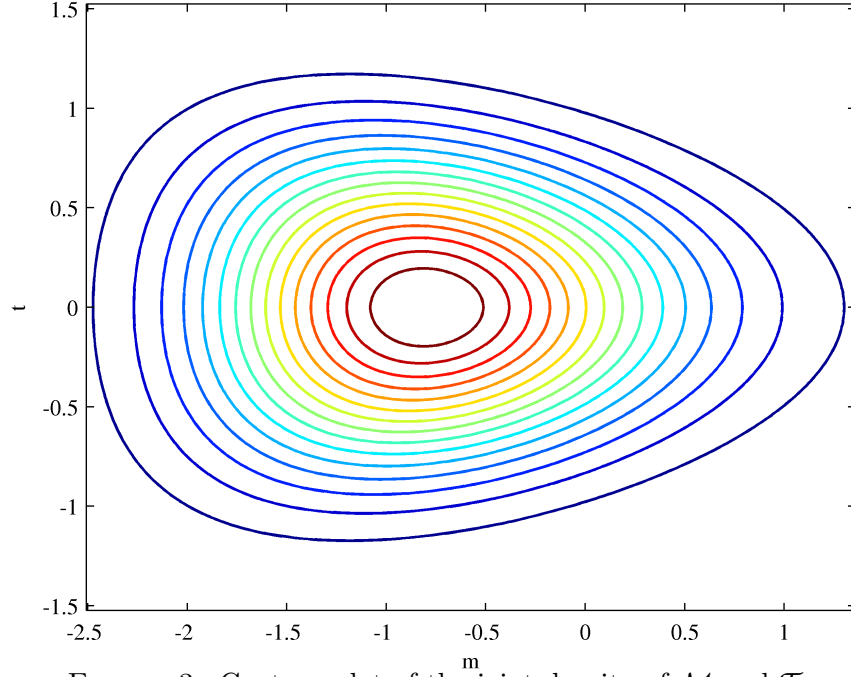
$$\varsigma_m(x, y) = (I - P_0 B_m P_0)^{-1}(x, y)$$

and, for $t, m \in \mathbb{R}$, define

$$\Psi_{t, m}(x, y) = 2^{1/3} \psi_{t, m}(2^{1/3}x) \psi_{-t, m}(2^{1/3}y)$$

and

$$\gamma(t, m) = 2^{1/3} \int_0^\infty dx \int_0^\infty dy \psi_{-t, 4^{-1/3}m}(2^{1/3}x) \varsigma_m(x, y) \psi_{t, 4^{-1/3}m}(2^{1/3}y).$$

FIGURE 3. Contour plot of the joint density of \mathcal{M} and \mathcal{T} .

Theorem 4.2 ([MFQR13], Theorem 2). *The joint density $f(t, m)$ of \mathcal{T} and \mathcal{M} is given by*

$$(4.14) \quad \begin{aligned} f(t, m) &= \gamma(t, 4^{1/3}m) F_{\text{GOE}}(4^{1/3}m) \\ &= \det(I - P_0 B_{4^{1/3}m} P_0 + P_0 \Psi_{t,m} P_0) - F_{\text{GOE}}(4^{1/3}m). \end{aligned}$$

To see where the second equality in (4.14) comes from, observe that $\gamma(t, 4^{1/3}m)$ equals the trace of the operator $(I - P_0 B_{4^{1/3}m} P_0)^{-1} P_0 \Psi_{t,m} P_0$ and that $\Psi_{t,m}$ is a rank one operator. The identity now follows that from the general fact that for two operators A and B such that B is rank one, one has $\det(I - A + B) = \det(I - A)[1 + \text{tr}((I - A)^{-1}B)]$.

Integrating over m one obtains a formula for the probability density $f_{\text{end}}(t)$ of \mathcal{T} . Unfortunately, it does not appear that the resulting integral can be calculated explicitly, so the best formula one has is

$$f_{\text{end}}(t) = \int_{-\infty}^{\infty} dm f(t, m).$$

One can readily check nevertheless that $f_{\text{end}}(t)$ is symmetric in t . The second formula for $f(t, m)$ is suitable for numerical computations, using the numerical scheme and Matlab toolbox developed by Bornemann in [Bor10a; Bor10b] for the computation of Fredholm determinants. Figure 3 shows a contour plot of the joint density of \mathcal{M} and \mathcal{T} , while Figure 4 shows a plot of the marginal \mathcal{T} density.

As we mentioned, interest in this problem dates back at least two decades. In particular, there has been a resurgence of interest in the last couple of years. An alternative way to obtain the Airy₂ process is as a limit in large N of the top path in a system of N non-intersecting random walks, or Brownian motions, the so called vicious walkers [Fis84] (this is of course related to the setting presented in Section 1.5.2). [SMCRF08; Fei09; RS10; RS11] obtained various expressions for the joint distributions of \mathcal{M} and \mathcal{T} in such a system at finite N . [FMS11] obtained the F_{GOE} distribution from large N asymptotics non-rigorously, and furthermore made connections between these problems and Yang-Mills theory. But for several years people were not able to perform asymptotic analysis on the formulas obtained for \mathcal{T} at finite N .

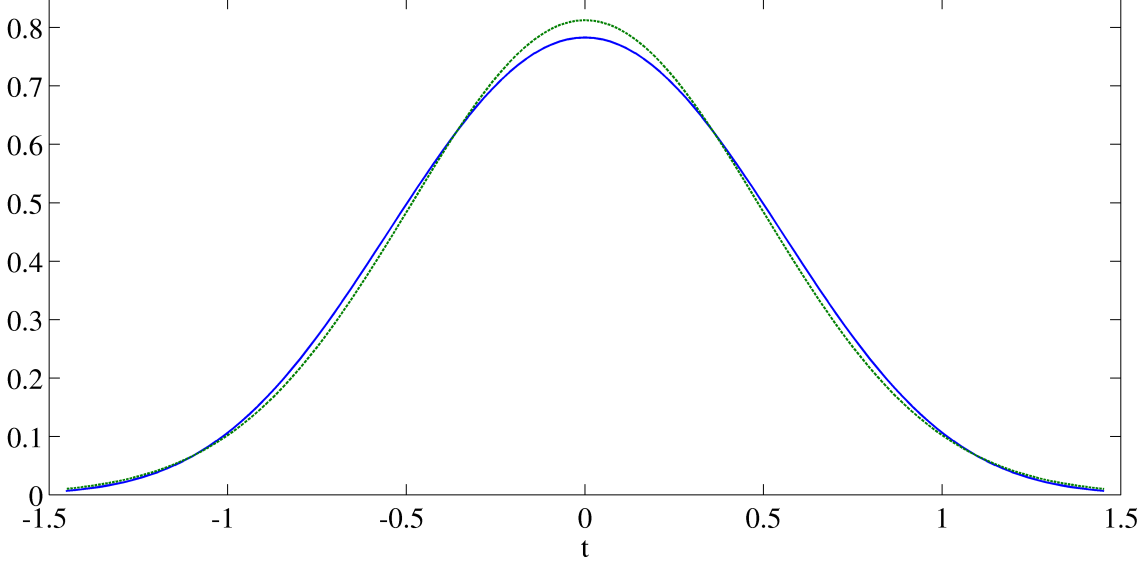


FIGURE 4. Plot of the density of \mathcal{T} compared with a Gaussian density with the same variance 0.2409 (dashed line). The excess kurtosis $\mathbb{E}(\mathcal{T}^4)/\mathbb{E}(\mathcal{T}^2)^2 - 3$ is -0.2374 .

After [MFQR13] appeared, Schehr [Sch12] succeeded in extracting asymptotics from the vicious walkers formula, and obtained an alternative formula for $f(t, m)$. His formula is given as follows. The Painlevé II equation (1.10)-(1.11) has a Lax pair formulation

$$(4.15) \quad \frac{\partial}{\partial \zeta} \Phi = A\Phi, \quad \frac{\partial}{\partial s} \Phi = B\Phi$$

for a two-dimensional vector $\Phi = \Phi(\zeta, s)$, where the 2×2 matrices $A = A(\zeta, s)$ and $B = B(\zeta, s)$ are given by

$$A(\zeta, s) = \begin{pmatrix} 4\zeta q & 4\zeta^2 + s + 2q^2 + 2q' \\ -4\zeta^2 - s - 2q^2 + 2q' & -4\zeta q \end{pmatrix} \quad \text{and} \quad B(\zeta, s) = \begin{pmatrix} q & \zeta \\ -\zeta & -q \end{pmatrix}.$$

The compatibility of this overdetermined system implies that $q(s)$ solves Painlevé II. Now let $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ be the unique solution of (4.15) satisfying

$$\Phi_1(\zeta; s) = \cos\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}), \quad \Phi_2(\zeta; s) = -\sin\left(\frac{4}{3}\zeta^3 + s\zeta\right) + O(\zeta^{-1}),$$

as $\zeta \rightarrow \pm\infty$ for $s \in \mathbb{R}$. The formula of [Sch12] is

$$(4.16) \quad \gamma(t, m) = \frac{16}{\pi^2} \langle h_{4^{2/3}t}, h_{-4^{2/3}t} \rangle_{L^2(m, \infty)}$$

where

$$h_t(x) = \int_0^\infty d\zeta \zeta \Phi_2(\zeta, x) e^{-t\zeta^2}.$$

Although Schehr's argument is non-rigorous, a later paper of Baik, Liechty, and Schehr [BLS12] proved directly the equivalence of the formula of [Sch12] and (4.14), thus establishing the validity of (4.16) based on Theorem 4.2.

Before turning to the tail behavior of \mathcal{T} , let us briefly explain how the uniqueness of the maximizer of $\mathcal{A}_2(t) - t^2$ can be established directly from the argument we described above. In the derivation of the formula we assumed that the maximum of $\mathcal{A}_2(t) - t^2$ is obtained at a unique point. However, it is not necessary to do this. In fact, if one follows the argument without this assumption, one ends up with a formula for what is in

principle a super-probability density, i.e. a non-negative function $f(t, m)$ on $\mathbb{R} \times \mathbb{R}$ with $\int_{\mathbb{R} \times \mathbb{R}} dm dt f(t, m) \geq 1$, and in fact one can see from the argument that

$$\int_{\mathbb{R} \times \mathbb{R}} dm dt f(t, m) = \text{expected number of maxima of } \mathcal{A}_2(t) - t^2.$$

Recall that from (4.1) that the distribution of \mathcal{M} is given by a scaled version of F_{GOE} . A non-trivial computation (see Section 3 of [MFQR13]) gives

$$\int_{-\infty}^{\infty} dt f(t, m) = 4^{1/3} F'_{\text{GOE}}(4^{1/3} m).$$

This shows that $f(t, m)$ has total integral 1, which can only be true if the maximum is unique almost surely, since the global maximum is attained at at least one point.

We mentioned earlier the conjecture that \mathcal{T} should have tails which decay like e^{-ct^3} (see e.g. [HHZ95]). This can be proved using the techniques described in this review:

Theorem 4.3 ([QR12b; Sch12; BLS12; CH11]). *There is a $c > 0$ such that for every $\kappa > \frac{32}{3}$ and large enough t ,*

$$e^{-\kappa t^3} \leq \mathbb{P}(|\mathcal{T}| > t) \leq ce^{-\frac{4}{3}t^3 + 2t^2 + \mathcal{O}(t^{3/2})}.$$

[CH11] had obtained the e^{-ct^3} decay for some $c > 0$. The statement we included here is the one appearing in [QR12b]. In fact, Schehr's formula and its validation in [BLS12] later yielded a lower bound that matches the $e^{-\frac{4}{3}t^3}$ behavior of the upper bound, so we know now that $\frac{4}{3}$ is the correct exponent. A precise asymptotic expansion of $\mathbb{P}(|\mathcal{T}| > t)$ based on that formula has recently been obtained in [BL13b]. The reason why [QR12b] obtained a slightly worse lower bound is technical, and arises from the fact that the explicit formula (4.14) for $f(t, m)$ is not useful for providing a lower bound, and instead one needs to use a different argument. On the other hand, the upper bound can be obtained directly from (4.14). In fact, the second formula expresses this joint density as the difference of two Fredholm determinants, so we may use (2.9) to estimate the difference, and then all that remains is to show that this estimate can be integrated in m . See [QR12b] for more details.

4.3. Local behavior of Airy_1 . As we mentioned, the boundary value kernel formulas introduced in Section 3.1 are better adapted than the standard extended kernel formulas to study short range properties of the processes. An interesting application is the following:

Theorem 4.4 ([QR12a], Theorem 2). *The Airy_1 process \mathcal{A}_1 and the Airy_2 process \mathcal{A}_2 have versions with Hölder continuous paths with exponent $\frac{1}{2} - \delta$ for any $\delta > 0$.*

Continuity was known for \mathcal{A}_2 (see Theorem 1.1) but not for \mathcal{A}_1 . The Hölder $\frac{1}{2}-$ continuity for \mathcal{A}_2 also follows from the work of Corwin and Hammond [CH11]. Their proof is based on a certain Brownian Gibbs property for the Airy_2 line ensemble (an infinite collection of continuous, non-intersecting paths, the top line of which is \mathcal{A}_2), and as such it cannot be extended to Airy_1 , given that no analog of the Airy_2 line ensemble is known in the flat case. This regularity is expected to hold in fact for all the Airy processes in view of the fact that they are believed to look locally like a Brownian motion (see Section 1.3). Analogous results have recently become available for the solutions of the KPZ equation at finite times with certain initial conditions [Hai13; QR11; CH12].

The proof of Theorem 4.4 is based on an application of a suitable version of the Kolmogorov criterion. In the Airy_1 case, it involves studying a truncated version of the process, $\mathcal{A}_1^M(t) = \mathcal{A}_1(t)\mathbf{1}_{|\mathcal{A}_1(t)| \leq M} + M\mathbf{1}_{\mathcal{A}_1(t) > M} - M\mathbf{1}_{\mathcal{A}_1(t) < -M}$ and then proving the following estimate: for fixed $\delta > 0$, there is a $t_0 \in (0, 1)$ and an $n_0 \in \mathbb{N}$ such that for $0 < t < t_0$, $n \geq n_0$ and $M = (3 \log(t^{-(1+n)}))^{1/3}$ we have

$$\mathbb{E}\left([\mathcal{A}_1^M(t) - \mathcal{A}_1^M(0)]^{2n}\right) \leq ct^{1+(1-\delta)n}$$

where the constant $c > 0$ is independent of δ , n_0 and t_0 . The proof of this estimate can be reduced to obtaining a suitable estimate on the difference

$$|\det(I - B_0 + \bar{P}_a e^{t\Delta} \bar{P}_b e^{-t\Delta} B_0) - \det(I - B_0 + \bar{P}_a B_0)|$$

for $b \geq a \geq -M$. An important technical problem is that the kernels appearing inside these determinants are not trace class, so one needs to conjugate appropriately. We refer to [QR12a] for the details. The argument for Airy_2 is similar.

As we mentioned, the Airy processes are expected to look locally like a Brownian motion. In this direction, it can be shown using the boundary value kernel formulas that the finite dimensional distributions of the Airy_1 process converge under diffusive scaling to those of a Brownian motion. The same result was proved earlier by Hägg [Häg08] for Airy_2 using different techniques. In fact, for Airy_2 a stronger statement is now available (Corwin and Hammond [CH11]), namely that it is locally absolutely continuous with respect to Brownian motion.

Theorem 4.5 ([QR12a], Theorem 3). *For any fixed $s \in \mathbb{R}$, let $B_\varepsilon(\cdot)$ be defined by $B_\varepsilon(t) = \varepsilon^{-1/2}(\mathcal{A}_1(s + \varepsilon t) - \mathcal{A}_1(s))$, $t > 0$. Then $B_\varepsilon(\cdot)$ converges to Brownian motion in the sense of convergence of finite dimensional distributions. The same holds for $\tilde{B}_\varepsilon(\cdot)$ defined by $\tilde{B}_\varepsilon(t) = B_\varepsilon(-t)$, $t > 0$.*

The proof of this result follows from an explicit computation of

$$\mathbb{P}(\mathcal{A}_1(\varepsilon t_1) \leq x + \sqrt{\varepsilon} y_1, \dots, \mathcal{A}_1(\varepsilon t_n) \leq x + \sqrt{\varepsilon} y_n \mid \mathcal{A}_1(0) = x)$$

and its limit as $\varepsilon \rightarrow 0$, see [QR12a] for the details. The same proof works for the Airy_2 process and, in view of (3.4), it should be simple to adapt it to the $\text{Airy}_{2 \rightarrow 1}$ process.

4.4. Marginals of $\text{Airy}_{2 \rightarrow 1}$. The last application of the results of Section 3.2 that we will discuss is a proof of the conjecture (1.34) that the marginals of the $\text{Airy}_{2 \rightarrow 1}$ process can be obtained from a variational problem for $\mathcal{A}_2(t) - t^2$ on a half-line. The result is the following:

Theorem 4.6 ([QR13], Theorem 1). *Fix $\alpha \in \mathbb{R}$. For every $m \in \mathbb{R}$,*

$$\mathbb{P}\left(\sup_{t \leq \alpha} (\mathcal{A}_2(t) - t^2) \leq m - \min\{0, \alpha\}^2\right) = \mathbb{P}(\mathcal{A}_{2 \rightarrow 1}(\alpha) \leq m).$$

The right hand side can be expressed in terms of a Fredholm determinant. Define the *crossover distributions* $G_\alpha^{2 \rightarrow 1}$, for $\alpha \in \mathbb{R}$, as

$$G_\alpha^{2 \rightarrow 1}(m) = \mathbb{P}(\mathcal{A}_{2 \rightarrow 1}(\alpha) \leq m).$$

We claim that

$$(4.17) \quad G_\alpha^{2 \rightarrow 1}(m) = \det(I - P_m K_\alpha P_m),$$

where $K_\alpha = K_\alpha^1 + K_\alpha^2$ and the kernels K_α^1 and K_α^2 are given by

$$K_\alpha^1(x, y) = \int_0^\infty d\lambda e^{2\alpha\lambda} \text{Ai}(x - \lambda + \max\{0, \alpha\}^2) \text{Ai}(y + \lambda + \max\{0, \alpha\}^2)$$

and

$$K_\alpha^2(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda + \max\{0, \alpha\}^2) \text{Ai}(y + \lambda + \max\{0, \alpha\}^2).$$

As noted in Appendix A of [BFS08b], the kernel $K_{2 \rightarrow 1}^{\text{ext}}$ defined in (1.24) can be expressed in terms of Airy functions:

$$K_{2 \rightarrow 1}^{\text{ext}}(s, t; x, y) = L_0(s, x; t, y) + e^{2t^3/3 - 2s^3/3 + t\tilde{y} - s\tilde{x}} [L_1 + L_2](s, x; t, y),$$

where

$$\begin{aligned} L_0(s, x; t, y) &= -e^{(s-t)\Delta}(\tilde{x}, \tilde{y}) = -\frac{1}{\sqrt{4\pi(t-s)}} e^{-(\tilde{x}-\tilde{y})^2/4(t-s)}, \\ L_1(s, x; t, y) &= \int_0^\infty d\lambda e^{\lambda(s+t)} \text{Ai}(\hat{x} - \lambda) \text{Ai}(\hat{y} + \lambda), \\ L_2(s, x, t, y) &= \int_0^\infty d\lambda e^{\lambda(t-s)} \text{Ai}(\hat{x} + \lambda) \text{Ai}(\hat{y} + \lambda) \end{aligned}$$

with $\tilde{x} = x - s^2 \mathbf{1}_{s \leq 0}$, $\tilde{y} = y - t^2 \mathbf{1}_{t \leq 0}$, $\hat{x} = x + s^2 \mathbf{1}_{s \geq 0}$ and $\hat{y} = y + t^2 \mathbf{1}_{t \geq 0}$. Using this for $s = t = \alpha$ it is straightforward to check that $K_{2 \rightarrow 1}^{\text{ext}}(t, \cdot; t, \cdot)$ is just a conjugation of the kernel K_α , and (4.17) follows.

The fact that $G_\alpha^{2 \rightarrow 1}$ crosses over between the GUE and GOE distributions is of course a particular case of the crossover property of the $\text{Airy}_{2 \rightarrow 1}$ process, but can be easily obtained from (4.17) as well (see the discussion after Theorem 1 in [QR13]).

The proof of Theorem 4.6 is similar to (and, in fact, somewhat simpler than) the proof of (1.25). Basically, one applies Theorem 3.1 and the cyclic property of determinants to compute the desired probability as

$$\lim_{L \rightarrow \infty} \mathbb{P}(\mathcal{A}_2(t) \leq g(t) \text{ for } t \in [-L, \alpha]) = \lim_{L \rightarrow \infty} \det \left(I - K_{\text{Ai}} + e^{(\alpha+L)H} K_{\text{Ai}} \Theta_{[-L, \alpha]}^g K_{\text{Ai}} \right)$$

with $g(t) = t^2 + \bar{m}$ and $\bar{m} = m - \min\{0, \alpha\}^2$. An argument similar to the one used in Section 4.1 (applying the Baker-Campbell-Hausdorff formula and later checking the result rigorously, plus some asymptotic analysis to show that an error term goes to 0 in trace class norm as $L \rightarrow \infty$) yields

$$\begin{aligned} (4.18) \quad \mathbb{P} \left(\sup_{t \leq \alpha} (\mathcal{A}_2(t) - t^2) \leq m - \min\{0, \alpha\}^2 \right) \\ = \det \left(I - K_{\text{Ai}} P_{\bar{m} + \alpha^2} K_{\text{Ai}} - K_{\text{Ai}} e^{\alpha \xi} \varrho_{\bar{m} + \alpha^2} e^{-\alpha \xi} \bar{P}_{\bar{m} + \alpha^2} K_{\text{Ai}} \right). \end{aligned}$$

Since $K_{\text{Ai}} = B_0 P_0 B_0$ and $B_0^2 = I$ we have by the cyclic property of determinants that the right hand side of (4.18) equals

$$\det \left(I - P_0 B_0 P_{\bar{m} + \alpha^2} B_0 P_0 - P_0 B_0 e^{\alpha \xi} \varrho_{\bar{m} + \alpha^2} e^{-\alpha \xi} \bar{P}_{\bar{m} + \alpha^2} B_0 P_0 \right).$$

Shifting the variables in the last determinant by $-m$ we deduce that

$$\mathbb{P} \left(\sup_{t \geq \alpha} (\mathcal{A}_2(t) - t^2) \leq \bar{m} \right) = \det(I - P_m E_1 P_m - P_m E_2 P_m),$$

where

$$E_1(x, y) = \int_{-\infty}^{\bar{m} + \alpha^2} d\lambda \text{Ai}(x - m + 2\bar{m} + 2\alpha^2 - \lambda) e^{-2(\lambda - \bar{m} - \alpha^2)\alpha} \text{Ai}(y - m + \lambda)$$

and

$$E_2(x, y) = \int_{\bar{m} + \alpha^2}^{\infty} d\lambda \text{Ai}(x - m + \lambda) \text{Ai}(y - m + \lambda).$$

Shifting λ by $\bar{m} + \alpha^2$ in both integrals and changing λ to $-\lambda$ shows that $E_1(x, y) = K_\alpha^1(y, x)$ and $E_2 = K_\alpha^2$, whence the equality in Theorem 4.6 follows since $E_1^* = K_\alpha^1$ and $E_2^* = K_\alpha^2$.

REFERENCES

- [AS64] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Vol. 55. National Bureau of Standards Applied Mathematics Series, 1964, pp. xiv+1046.
- [AKQ12a] T. Alberts, K. Khanin, and J. Quastel. *Intermediate Disorder Regime for 1+1 Dimensional Directed Polymers*. 2012. arXiv:1202.4398.
- [AKQ12b] T. Alberts, K. Khanin, and J. Quastel. *The continuum directed random polymer*. 2012. arXiv:1202.4403.
- [ACQ11] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.* 64.4 (2011), pp. 466–537.
- [AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*. Vol. 118. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2010, pp. xiv+492.
- [AD11] A. Auffinger and M. Damron. *A simplified proof of the relation between scaling exponents in first-passage percolation*. 2011. arXiv:1109.0523.
- [AD12] A. Auffinger and M. Damron. *The scaling relation $\chi = 2\xi - 1$ for directed polymers in a random environment*. 2012. arXiv:1211.0992.
- [BL13a] J. Baik and Z. Liu. *On the average of the Airy process and its time reversal*. 2013. arXiv:1308.1070.
- [BBD08] J. Baik, R. Buckingham, and J. DiFranco. Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function. *Comm. Math. Phys.* 280.2 (2008), pp. 463–497.
- [BDJ99] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* 12.4 (1999), pp. 1119–1178.
- [BFP10] J. Baik, P. L. Ferrari, and S. Péché. Limit process of stationary TASEP near the characteristic line. *Comm. Pure Appl. Math.* 63.8 (2010), pp. 1017–1070.
- [BLS12] J. Baik, K. Liechty, and G. Schehr. *On the joint distribution of the maximum and its position of the Airy_2 process minus a parabola*. 2012. arXiv:1205.3665.
- [BR00] J. Baik and E. M. Rains. Limiting distributions for a polynuclear growth model with external sources. *J. Stat. Phys.* 100.3-4 (2000), pp. 523–541.
- [BR01] J. Baik and E. M. Rains. Symmetrized random permutations. In: *Random matrix models and their applications*. Vol. 40. Math. Sci. Res. Inst. Publ. Cambridge: Cambridge Univ. Press, 2001, pp. 1–19.
- [BG97] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.* 183.3 (1997), pp. 571–607.
- [Bor10a] F. Bornemann. On the numerical evaluation of distributions in random matrix theory: a review. *Markov Process. Related Fields* 16.4 (2010), pp. 803–866.
- [Bor10b] F. Bornemann. On the numerical evaluation of Fredholm determinants. *Math. Comp.* 79.270 (2010), pp. 871–915.
- [BG12] A. Borodin and V. Gorin. *Lectures on integrable probability*. 2012. arXiv:1212.3351.
- [Bor11] A. Borodin. Determinantal point processes. In: *The Oxford Handbook of Random Matrix Theory*. Oxford University Press, 2011.
- [BC11] A. Borodin and I. Corwin. *Macdonald Processes*. To appear in Prob. Theory Related Fields. 2011. arXiv:1111.4408.
- [BCF12] A. Borodin, I. Corwin, and P. Ferrari. *Free energy fluctuations for directed polymers in random media in 1+1 dimension*. To appear in Comm. Pure. Appl. Math. 2012. arXiv:1204.1024.
- [BCR] A. Borodin, I. Corwin, and D. Remenik. *Multiplicative functionals on ensembles of non-intersecting paths*. To appear in Ann. Inst. Henri Poincaré Probab. Stat.

- [BCR13] A. Borodin, I. Corwin, and D. Remenik. Log-Gamma polymer free energy fluctuations via a Fredholm determinant identity. *Comm. Math. Phys.* (2013), pp. 1–18. DOI: [10.1007/s00220-013-1750-x](https://doi.org/10.1007/s00220-013-1750-x).
- [BF08] A. Borodin and P. L. Ferrari. Large time asymptotics of growth models on space-like paths. I. PushASEP. *Electron. J. Probab.* 13 (2008), no. 50, 1380–1418.
- [BFP07] A. Borodin, P. L. Ferrari, and M. Prähofer. Fluctuations in the discrete TASEP with periodic initial configurations and the Airy_1 process. *Int. Math. Res. Pap. IMRP* (2007), Art. ID rpm002, 47.
- [BFPS07] A. Borodin, P. L. Ferrari, M. Prähofer, and T. Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.* 129.5-6 (2007), pp. 1055–1080.
- [BFS08a] A. Borodin, P. L. Ferrari, and T. Sasamoto. Large time asymptotics of growth models on space-like paths. II. PNG and parallel TASEP. *Comm. Math. Phys.* 283.2 (2008), pp. 417–449.
- [BFS08b] A. Borodin, P. L. Ferrari, and T. Sasamoto. Transition between Airy_1 and Airy_2 processes and TASEP fluctuations. *Comm. Pure Appl. Math.* 61.11 (2008), pp. 1603–1629.
- [BFS09] A. Borodin, P. L. Ferrari, and T. Sasamoto. Two speed TASEP. *J. Stat. Phys.* 137.5-6 (2009), pp. 936–977.
- [BO00] A. Borodin and G. Olshanski. Distributions on partitions, point processes, and the hypergeometric kernel. *Comm. Math. Phys.* 211.2 (2000), pp. 335–358.
- [BR05] A. Borodin and E. M. Rains. Eynard-Mehta theorem, Schur process, and their Pfaffian analogs. *J. Stat. Phys.* 121.3-4 (2005), pp. 291–317.
- [BS02] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Second Edition. Probability and its Applications. Birkhäuser Verlag, 2002, pp. xvi+672.
- [BL13b] T. Bothner and K. Liechty. Tail decay for the distribution of the endpoint of a directed polymer. *Nonlinearity* 26.5 (2013), p. 1449.
- [Cha12] S. Chatterjee. The universal relation between scaling exponents in first-passage percolation. To appear in *Annals of Math.* (2012).
- [CQ11] I. Corwin and J. Quastel. *Renormalization fixed point of the KPZ universality class*. 2011. arXiv:[1103.3422](https://arxiv.org/abs/1103.3422).
- [CFP10] I. Corwin, P. L. Ferrari, and S. Péché. Limit processes for TASEP with shocks and rarefaction fans. *J. Stat. Phys.* 140.2 (2010), pp. 232–267.
- [CFP12] I. Corwin, P. L. Ferrari, and S. Péché. Universality of slow decorrelation in KPZ growth. *Ann. Inst. Henri Poincaré Probab. Stat.* 48.1 (2012), pp. 134–150.
- [CH11] I. Corwin and A. Hammond. *Brownian Gibbs property for Airy line ensembles*. To appear in *Inventiones Mathematicae*. 2011. arXiv:[1108.2291](https://arxiv.org/abs/1108.2291).
- [CH12] I. Corwin and A. Hammond. *The H-Brownian Gibbs property of the KPZ line ensemble*. In preparation. 2012.
- [CLW] I. Corwin, Z. Liu, and D. Wang. In preparation.
- [COSZ11] I. Corwin, N. O’Connell, T. Seppäläinen, and N. Zygouras. *Tropical Combinatorics and Whittaker functions*. To appear in *Duke Math. J.* 2011. arXiv:[1110.3489](https://arxiv.org/abs/1110.3489).
- [CQR13] I. Corwin, J. Quastel, and D. Remenik. Continuum statistics of the airy_2 process. *Comm. Math. Phys.* 317.2 (2013), pp. 347–362.
- [EM98] B. Eynard and M. L. Mehta. Matrices coupled in a chain. I. Eigenvalue correlations. *J. Phys. A* 31.19 (1998), pp. 4449–4456.

- [Fei09] T. Feierl. The height and range of watermelons without wall. In: *Combinatorial Algorithms*. Vol. 5874. Lecture Notes in Computer Science. Springer Berlin / Heidelberg, 2009, pp. 242–253.
- [Fer08] P. L. Ferrari. Slow decorrelations in Kardar–Parisi–Zhang growth. *J. Stat. Mech.* 2008.07 (2008), P07022.
- [FF13] P. L. Ferrari and R. Frings. On the spatial persistence for airy processes. *J. Stat. Mech.* 2013.02 (2013), P02001.
- [FS05] P. L. Ferrari and H. Spohn. A determinantal formula for the GOE Tracy–Widom distribution. *J. Phys. A* 38.33 (2005), pp. L557–L561.
- [Fis84] M. E. Fisher. Walks, walls, wetting, and melting. *J. Stat. Phys.* 34 (5 1984), pp. 667–729.
- [FNH99] P. J. Forrester, T. Nagao, and G. Honner. Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges. *Nucl. Phys. B* 553.3 (1999), pp. 601–643.
- [FMS11] P. J. Forrester, S. N. Majumdar, and G. Schehr. Non-intersecting brownian walkers and Yang–Mills theory on the sphere. *Nucl. Phys. B* 844.3 (2011), pp. 500–526.
- [GV85] I. Gessel and G. Viennot. Binomial determinants, paths, and hook length formulae. *Adv. in Math.* 58.3 (1985), pp. 300–321.
- [Gro89] P. Groeneboom. Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields* 81.1 (1989), pp. 79–109.
- [Häg08] J. Hägg. Local Gaussian fluctuations in the Airy and discrete PNG processes. *Ann. Probab.* 36.3 (2008), pp. 1059–1092.
- [Hai13] M. Hairer. Solving the KPZ equation. *Ann. of Math. (2)* 178.2 (2013), pp. 559–664.
- [HHZ95] T. Halpin-Healy and Y.-C. Zhang. Kinetic roughening phenomena, stochastic growth, directed polymers and all that. *Phys. Rep.* 254.4-6 (1995), pp. 215–414.
- [Joh00] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.* 209.2 (2000), pp. 437–476.
- [Joh03] K. Johansson. Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.* 242.1-2 (2003), pp. 277–329.
- [Joh06] K. Johansson. Random matrices and determinantal processes. In: *Mathematical statistical physics*. Elsevier B. V., Amsterdam, 2006, pp. 1–55.
- [KPZ86] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamical scaling of growing interfaces. *Phys. Rev. Lett.* 56.9 (1986), pp. 889–892.
- [KM59] S. Karlin and J. McGregor. Coincidence probabilities. *Pacific J. Math.* 9 (1959), pp. 1141–1164.
- [Koc92] H. von Koch. Sur les déterminants infinis et les équations différentielles linéaires. *Acta Math.* 16.1 (1892), pp. 217–295.
- [Lin73] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.* 5 (1973), pp. 85–90.
- [Mac94] A. M. S. Macêdo. Universal parametric correlations at the soft edge of the spectrum of random matrix ensembles. *Europhysics Letters* 26.9 (1994), p. 641.
- [Meh91] M. L. Mehta. *Random matrices*. Second. Boston, MA: Academic Press Inc., 1991, pp. xviii+562.
- [MP92] M. Mézard and G. Parisi. A variational approach to directed polymers. *J. Phys. A* 25.17 (1992), pp. 4521–4534.
- [MFQR13] G. Moreno Flores, J. Quastel, and D. Remenik. Endpoint distribution of directed polymers in $1 + 1$ dimensions. *Comm. Math. Phys.* 317.2 (2013), pp. 363–380.
- [O’C12] N. O’Connell. Directed polymers and the quantum Toda lattice. *Ann. Probab.* 40.2 (2012), pp. 437–458.

- [Pim12] L. P. R. Pimentel. *On the location of the maximum of a continuous stochastic process*. 2012. arXiv:[1207.4469](#).
- [PS02] M. Prähofer and H. Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Stat. Phys.* 108.5-6 (2002), pp. 1071–1106.
- [PS11] S. Prolhac and H. Spohn. The one-dimensional KPZ equation and the Airy process. *J. Stat. Mech. Theor. Exp.* 2011.03 (2011), P03020.
- [QR11] J. Quastel and D. Remenik. Local Brownian property of the narrow wedge solution of the KPZ equation. *Electron. Comm. Probab.* 16 (2011), pp. 712–719.
- [QR12a] J. Quastel and D. Remenik. Local behavior and hitting probabilities of the Airy_1 process. *Probab. Theory Related Fields* (2012), pp. 1–30. DOI: [10.1007/s00440-012-0466-8](#).
- [QR12b] J. Quastel and D. Remenik. *Tails of the endpoint distribution of directed polymers*. To appear in Ann. Inst. Henri Poincaré Probab. Stat. 2012. arXiv:[1203.2907](#).
- [QR13] J. Quastel and D. Remenik. Supremum of the airy_2 process minus a parabola on a half line. *J. Stat. Phys.* 150.3 (2013), pp. 442–456.
- [RS10] J. Rambeau and G. Schehr. Extremal statistics of curved growing interfaces in 1+1 dimensions. *EPL (Europhysics Letters)* 91.6 (2010).
- [RS11] J. Rambeau and G. Schehr. Distribution of the time at which n vicious walkers reach their maximal height. *Phys. Rev. E* 83 (6 2011), p. 061146.
- [Sas05] T. Sasamoto. Spatial correlations of the 1D KPZ surface on a flat substrate. *Journal of Physics A: Mathematical and General* 38.33 (2005), p. L549.
- [Sch12] G. Schehr. *Extremes of N vicious walkers for large N : application to the directed polymer and KPZ interfaces*. 2012. arXiv:[arXiv:1203.1658](#).
- [SMCRF08] G. Schehr, S. N. Majumdar, A. Comtet, and J. Randon-Furling. Exact distribution of the maximal height of p vicious walkers. *Phys. Rev. Lett.* 101.15 (2008), pp. 150601, 4.
- [Sim05] B. Simon. *Trace ideals and their applications*. Second. Vol. 120. Mathematical Surveys and Monographs. American Mathematical Society, 2005, pp. viii+150.
- [TS10] K. Takeuchi and M. Sano. Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals. *Phys. Rev. Lett.* 104 (23 2010), p. 230601.
- [TS12] K. Takeuchi and M. Sano. Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence. *J. Stat. Phys.* 147 (5 2012), pp. 853–890.
- [TW94] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* 159.1 (1994), pp. 151–174.
- [TW96] C. A. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* 177.3 (1996), pp. 727–754.
- [TW04] C. A. Tracy and H. Widom. Differential equations for Dyson processes. *Comm. Math. Phys.* 252.1-3 (2004), pp. 7–41.
- [TW06] C. A. Tracy and H. Widom. The Pearcey process. *Comm. Math. Phys.* 263.2 (2006), pp. 381–400.
- [TW07] C. A. Tracy and H. Widom. Nonintersecting Brownian excursions. *Ann. Appl. Probab.* 17.3 (2007), pp. 953–979.
- [TW08a] C. A. Tracy and H. Widom. A Fredholm determinant representation in ASEP. *J. Stat. Phys.* 132.2 (2008), pp. 291–300.
- [TW08b] C. A. Tracy and H. Widom. Integral formulas for the asymmetric simple exclusion process. *Comm. Math. Phys.* 279.3 (2008), pp. 815–844.
- [TW09] C. A. Tracy and H. Widom. Asymptotics in ASEP with step initial condition. *Comm. Math. Phys.* 290.1 (2009), pp. 129–154.
- [VS10] O. Vallée and M. Soares. *Airy functions and applications to physics*. Second. London: Imperial College Press, 2010, pp. x+202.

- [Wal86] J. B. Walsh. An introduction to stochastic partial differential equations. In: *École d'été de probabilités de Saint-Flour, XIV—1984*. Vol. 1180. Lecture Notes in Math. Berlin: Springer, 1986, pp. 265–439.
- [Wid04] H. Widom. On asymptotics for the Airy process. *J. Stat. Phys.* 115.3-4 (2004), pp. 1129–1134.

(J. Quastel) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO, ONTARIO, CANADA M5S 2E4

E-mail address: `quastel@math.toronto.edu`

(D. Remenik) DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CENTRO DE MODELAMIENTO MATEMÁTICO, UNIVERSIDAD DE CHILE, AV. BLANCO ENCALADA 2120, SANTIAGO, CHILE

E-mail address: `dremenik@dim.uchile.cl`